Lecture #9

April 9 2013

$$G = sP + s\frac{1}{n}ae^T + (1-s)\frac{1}{n}ee^T$$

$$= sP + \left[sa + (1-s)e\right]\frac{e^T}{n}$$

Sparse Raw Stochastic Hyper-link Matrix (But Reducible).

Completely Dense Rank-One Teleportation Matrix.

G = Completely Dense Stochastic Primitive Google Matrix.

$s \in [0, 1]$ = Scaling Parameter.

a = Binary Dangling Row Vector.

Iterative Equation (for Updating).

$$\omega_{k+1}^T = \omega_k^T G.$$

$$\omega^{*T} = \omega^{*T} G \longrightarrow \omega^{*} = \text{Fixpoint}$$

$\boxed{= \text{Page Rank Vector.}}$

$= \text{Stationary Row Vector.}$

Solve.

(A) Eigenvector problem for ω^T

$\omega^T = \omega^T G$ (Note $\alpha \omega^T = \alpha \omega^T G$)

$\omega^T e = 1.$

(B) Linear Homogeneous System.

$\omega^T (I - G) = 0^T$ (Note $\alpha \omega^T (I-G) = 0^T$)

$\omega^T e = 1.$ ω^T = Left-Null Vector of $I-G$

(A) Compute normalized dominant left-hand eigen vector of G.

{ Corresponding to dominant eigenvalue, $\lambda_1 = 1$. ($\because G$ = Stochastic)

(B) Compute normalized left-hand null vector of $I-G$.

$\omega^T e = 1.$ ← Normalization Equation.

$\Rightarrow \omega^T$ = Probability.

$\boxed{\text{POWER METHOD}}$

An iterative method for computing the dominant eigenvector (and eigenvalue).

(Usually, it is one of the slowest algorithms.)

$$\omega_{k+1}^T = \underbrace{s\omega_k^T P}_{\text{Sparse Update}} + \left[s\omega_k^T a + (1-s)\right]\frac{e^T}{n}$$

Note $\omega_k^T e = 1$.

Rank-One Update.

MATRIX-FREE COMPUTATION:

Google matrix G is never computed or stored.!

Storage $\begin{cases} \text{Sparse Matrix } P \\ \text{Dangling Vector } a \\ \text{PageRank Vector } \omega_k^T \end{cases}$

Asymptotic Convergence of the Power Method.

→ Ratio of the two "largest" (in magnitude) eigenvalues of the matrix.

$$\left|\frac{\lambda_2}{\lambda_1}\right|^k \longrightarrow 0$$

G = Stochastic $\Rightarrow \lambda_1 = 1$.

∴ Convergence depends only on $|\lambda_2|$

Since G = primitive, $|\lambda_2| < 1$.
(Irreducible + Aperiodic)

$$P' = P + a e^T/n$$

Spectrum of P' = $\sigma(P')$ =
$\{1, \mu_2, \dots, \mu_n\}$
$1 > |\mu_2| > \dots |\mu_n|$.

$G = sP^T + (1-s)ee^T/n$

Spectrum of $G = \sigma(G)$
$\{1, \lambda_2, \dots, \lambda_n\}$

(I) $\forall_{k\geq 2}$ $\lambda_k = s\mu_k$

(II) $|\mu_2| = 1$. a.s. (which occurs often due to reducibility of the web graph.

(III) $|\lambda_2| = s$ a.s.

$s = 0.85 \Rightarrow 0.85^{50} = 0.000296.$

$$\left.\begin{aligned} s^k &= \varepsilon > 0 \\ k \ln s &= \ln \varepsilon \\ k &= \frac{\ln \varepsilon}{\ln s} \end{aligned}\right\}$$ $k = 50$ 0.03% accuracy in PageRank.

$P' =$ Stochastic Matrix

$e = P'e$ $\therefore \begin{cases} 1 = \lambda_1 \\ e = 1^{st} \text{ eigenvector.} \end{cases}$

Let $Q = [e\ X]$

$$\Rightarrow Q^{-1} = \begin{bmatrix} y^T \\ Y^T \end{bmatrix}$$

$$Q^{-1}Q = \begin{bmatrix} y^T e & y^T X \\ Y^T e & Y^T X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$$

$$Q^{-1}P'Q = \begin{bmatrix} y^T e & y^T P'X \\ Y^T e & Y^T P'X \end{bmatrix} = \begin{bmatrix} 1 & y^T P'X \\ 0 & Y^T P'X \end{bmatrix}$$

$$Q^{-1}GQ = sQ^{-1}P'Q + (1-s)\,Q^{-1}e\,v^T Q \qquad \frac{e^T}{n} > 0$$

$$= s\begin{bmatrix} 1 & y^T P'X \\ 0 & Y^T P'X \end{bmatrix} + (1-s)\begin{bmatrix} 1 & v^T X \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & [\cdots] \\ 0 & sY^T P'X \end{bmatrix}$$

$Q^{-1}SQ \leftarrow$ Similarity Transformation.

$$\sigma(P') = \{1, \mu_2, \cdots, \mu_n\} = \sigma(Q^{-1}P'Q)$$

$$\Rightarrow \sigma(Y^T P' X) = \{\mu_2, \cdots, \mu_n\}$$

$$\Rightarrow \sigma(s Y^T P' X) = \{s\mu_2, \cdots, s\mu_n\}$$

$$\Rightarrow \sigma(Q^{-1} G Q) = \sigma(G)$$

$$= \{1, s\mu_2, \cdots, s\mu_n\}$$