Lecture #11

April 23 2013

Strategic Form Games.
(Normal-Form or Matrix Games).

(A) All participants act simultaneously and without knowledge of other player's actions (or thoughts/intentions etc.)

(B) Main Ingredients:
- The set of players (I)
- The strategies $(S_{i \in I})$
- The payoffs $(u_{i \in I})$.

(C) In general, we may also need
- The game form.
  (Captures order of play)
- Information set.
  (Models asymmetric information or incomplete information situations).

STRATEGIC FORM GAME:

A strategic form game is a triplet

$$\langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle$$

such that

- I is a finite set of players
- S_i is the set of actions (strategies) available to player $i \in I$.
- $s_i \in S_i$ is an action (strategy) for player i.
- $u_i : S \to R$ $\{ S = \Pi_i S_i \}$ is the payoff function of player i & S = the set of all strategy profiles.

Notation:

$$s_{-i} = [s_j]_{j \neq i}$$ Vectors of actions for all players except i.

$$S_{-i} = \Pi_{j \neq i} S_j$$ Set of strategy profiles for all players except i.

$$(s_i, s_{-i}) \in S$$ A STRATEGY PROFILE or OUTCOME.

$\boxed{\text{Best Response}}$

$$B_i(s_{-i}) \in \underset{s_i \in S_i}{\arg\max}\ u_i(s_i, s_{-i})$$

A strategy of i that maximizes his utility provided that other players have selected s_{-i}.

We would like for every one to choose their best responses and not deviate from it:

choose s^* s.t.

$$\forall_{i \in I}.\ B_i(s^*_{-i}) = s^*_i$$

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PARTNERSHIP GAME

Friend$_1$ \ Friend$_2$	Work Hard	Shirk
Work Hard	(2, 2) *	(-1, 1)
Shirk	(1, -1)	(0, 0) *

$B_1(-, WH) = WH$

$B_1(-, S) = S$

$B_2(WH, -) = WH$

$B_2(S, -) = S$

More Complex Game

Matching Penny

Matcher \ Mismatcher	H	T
H	(1, -1)	(-1, 1)
T	(-1, 1)	(1, -1)

ROCK-PAPER-SCISSORS (pp 73)

$P_1 \backslash P_2$

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Neither of these two games (both 0-sum) has a

PURE-STRATEGY-NASH-EQ.

DOMINANT STRATEGY.

A strategy $s_i \in S_i$ is dominant for player i if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$
$$\forall s_i' \in S_i \wedge \forall s_{-i} \in S_{-i}.$$

STRICTLY DOMINATED STRATEGY.

A strategy $s_i \in S_i$ is strictly dominated for player i if

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$
$$\exists s_i' \in S_i \wedge \forall s_{-i} \in S_{-i}.$$

WEAKLY DOMINATED STRATEGY

A strategy $s_i \in S_i$ is weakly dominated for player i if

$$\exists s_i' \in S_i$$
$$\forall s_{-i} \in S_{-i} \quad u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$
$$\wedge$$
$$\exists s_{-i} \in S_{-i} \quad u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

CKR + P (Common knowledge of Rationality & Payoffs)

$\Rightarrow$ Iterated Elimination of Strictly Dominated Strategies

$j := 0;$
for each $i \in I$
$S_i^0 := S_i;$
$j := j+1;$
loop
for each $i \in I$

$$S_i^j := \left\{ s_i \in S_i^{j-1} \;\middle|\; \nexists s_i' \in S_i^{j-1}\ \forall s_{-i} \in S_{-i}^{j-1}\ u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \right\}$$

$$S_i^\infty := \bigcap_{k=0}^{\infty} S_i^k;$$

$$S^\infty := \prod_i S_i^\infty$$

Dominant Strategy Equilibrium.

A strategy profile S^* is the dominant strategy equlibrium if for each player i,

$S_i^* =$ is a dominant strategy.

PRISONER'S DILEMMA

Prisoner 1 \ Prisoner 2	Confess	Don't confess	Suicide
Confess	(-2, -2)	(0, -3)	(-2, -10)
Don't Confess	(-3, 0)	(-1, -1)	(0, -10)
Suicide	(-10, -2)	(-10, 0)	(-10, -10)

- Suicide is dominated for both players
  ⇒ Eliminate Suicide.

- Don't Confess is dominated for both players
  ⇒ Eliminate Don't confess.

$$S^{\infty} = \{ (\text{Confess}, \text{confess}) \}$$

DOMINANT STRATEGY EQUILIBRIUM

(Follows from CKR).

MIXED STRATEGIES (NASH)

Σ_i = Probability Measures over the Pure Strategies S_i (for player i)

$\sigma_i \in \Sigma_i$ = MIXED STRATEGY of player i

$\sigma_i = (p_{i_1}, p_{i_2}, \dots, p_{ik})$ corresponding to i's strategies $s_{i_1}, s_{i_2}, \dots, s_{i_K}$

$Pr[s_{i_j} \in S_i \text{ is played}] = p_{ij}$

$p_{ij} \geq 0 \qquad \sum p_{ij} = 1.$

$\Sigma = \Pi_{i \in I} \Sigma_i$ = Mixed strategy profiles.

$\sigma \in \Sigma$

PLAYERS RANDOMIZE INDEPENDENTLY:

$$u_i(\sigma) = \int_S u_i(s)\, d\sigma(s)$$
$$= \sum p_{ij}\, u_i(s_{ij}, \sigma_{-i})$$

MIXED STRATEGY NASH EQ.

Defn:
A mixed strategy profile σ^* is a (mixed strategy) Nash Equilibrium, if

$$\forall_{i \in I}\ \forall_{\sigma_i \in \Sigma_i} \quad u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

NASH'S THEOREM:

Every finite game has a mixed strategy Nash Equilibrium.

Matching Pennies:

Mismatcher \ Matcher	H	T
H	(-1,1)	(1,-1)
T	(1,-1)	(-1,1)

It has a Unique Mixed Strategy Nash Equilibrium: $((1/2, 1/2), (1/2, 1/2))$

BoS (Battle of Sexes Game)

Male \ Female	Opera	Football
Opera	(1, 4)	(0, 0)
Football	(0, 0)	(4, 1)

Two Pure Nash Equilibria

+ One Mixed Nash Equilibrium

$$((4/5, 1/5), (1/5, 4/5))$$

KAKUTANI'S FIXED POINT THEOREM

Let $f : A \rightrightarrows A \; ; \; x \in A \mapsto f(x) \subseteq A$.
be a correspondence
satisfying the following conditions:

(A) A = Compact, convex and non-empty subset of a finite dimensional Euclidean space.

(B) $\forall_{x \in A} \; f(x) \neq \phi$

(C) $\forall_{x \in A} \; f(x)$ = Convex Set

(D) $f(x)$ has a closed graph

$$\{x^n, y^n\} \rightarrow \{x, y) \text{ with } y^n \in f(x^n) \Rightarrow y \in f(x).$$

Then f has a fixed point

$$\exists x^* \in A \quad x^* \in f(x^*) \quad \square$$

Corollary: Nash Mixed Strategy Equilibria:

$f \equiv$ Best responses ...