$\boxed{\text{LECTURE \# 9}}$

Completeness Thm

Gödel 1930: If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

(Show that any consistent set of formulas is satisfiable)

$\Downarrow$

Model exists!

$\mathcal{L}$ = Language (+ Signature, Σ)

c: Constant symbol

$\mathcal{L}_c$ = The result of adjoining c to $\mathcal{L}$. $\left\{ \therefore \mathcal{L}_c = \mathcal{L} \text{ if } c \text{ already occurs in } \mathcal{L}. \right.$

C = Set of constant symbols.

$\mathcal{L}C$: The language resulting from $\mathcal{L}$ by adjoining a set C of constants to $\mathcal{L}$.

= A CONSTANT EXPANSION OF $\mathcal{L}$.

α^c_z = Formula arising from α by replacing constant c with variable z.

$X^c_z = \{ \alpha^c_z \mid \alpha \in X \} \Rightarrow \left\{ \begin{matrix} c \text{ no longer} \\ \text{occurs in } X^c_z \end{matrix} \right\}$

CEL : $\boxed{\text{Constant Elimination Lemma.}}$

Suppose $X \vdash_{\mathcal{L}_c} \alpha$. Then $X^c_x \vdash_{\mathcal{L}} \alpha^c_x$, $\left\{ \begin{array}{l} \text{for almost all} \\ \text{variables } x. \end{array} \right.$ $\square$

Proof: By induction in $\vdash_{\mathcal{L}_c}$ ($\Rightarrow$ HW).

$\boxed{\text{Corollary 1}}$ **RCQ** $\boxed{\text{Rule of Constant Quantification}}$

$$\frac{X \vdash \alpha^c_x}{X \vdash \forall_x \alpha} \quad (c \notin X, \alpha, \text{etc.}) \qquad \square$$

$\boxed{\text{Corollary 2}}$ Conservative Expansion.

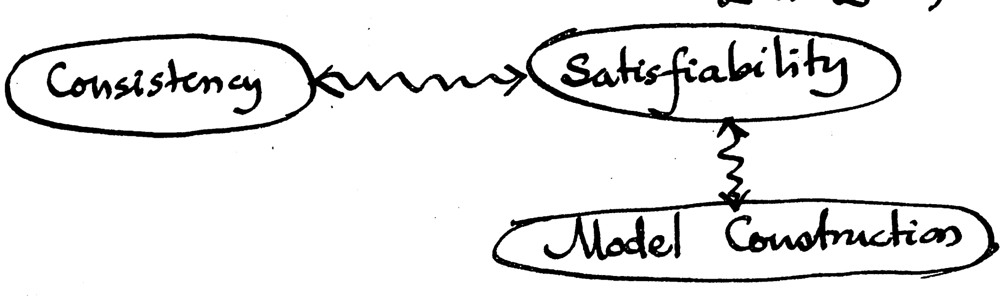
Let C be any set of constant symbols & $\mathcal{L}' = \mathcal{L}C$.

Then, for all $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$

$X \vdash_{\mathcal{L}} \alpha$ iff $X \vdash_{\mathcal{L}'} \alpha$. $\square$

$\boxed{\text{Abuse of Notation.}}$ ① $\vdash$ stands for derivability relation in $\mathcal{L}$ ... (and in every constant expansion $\mathcal{L}' = \mathcal{L}C$).

② $X \subseteq \mathcal{L}$; $X \nvdash \perp$ ≡ 'X is consistent' (No need to distinguish bet'n consistency of X w.r.t. $\mathcal{L}$ or $\mathcal{L}'$...)



Model Construction

For each variable x and each wff $\alpha \in \mathcal{L}$

$\Rightarrow$ Choose a constant $c_{x,\alpha}$ not occurring in $\mathcal{L}$ as follows:

(*) $\qquad \alpha^x := \neg \forall x\, \alpha \wedge \alpha^x_c \qquad c := c_{x,\alpha}$

Thus $\quad \neg \alpha^x := \exists x\, \neg \alpha \to \neg \alpha^x_c$ $\quad \left\{ \begin{array}{l} \text{Thus, constant } c \text{ is a} \\ \text{counter-example to} \\ \text{the validity of } \alpha. \end{array} \right.$

Lemma

Let

$T_{\mathcal{L}} := \{ \neg \alpha^x \mid \alpha \in \mathcal{L},\ x \in \mathrm{var} \}$, where α^x is defined by (*) & $X \subseteq \mathcal{L}$ is consistent.

Then $X \cup T_{\mathcal{L}}$ is consistent as well.

Proof: Suppose not. I.e. $X \cup T_{\mathcal{L}} \vdash \bot$.

Then $\exists_{n \geq 0} \exists\, \neg\alpha_0^{x_0}, \ldots, \neg\alpha_n^{x_n}$ (= wff's) s.t.

$$X \cup \{ \neg \alpha_i^{x_i} \mid i \leq n \} \vdash \bot$$

But $X' := X \cup \{ \neg \alpha_i^{x_i} \mid i < n \} \nvdash \bot$.

$x := x_n;\ \alpha := \alpha_n \qquad c = c_{x,\alpha}$.

$\therefore X' \cup \{ \neg \alpha^x \} \vdash \bot$

$\Rightarrow X' \vdash \alpha^x := \neg \forall x\, \alpha \wedge \alpha^x_c$

$\Rightarrow X' \vdash \neg \forall x\, \alpha$; $X' \vdash \alpha^x_c$; $X' \vdash \alpha^x_c \to \forall x\, \alpha$ (Ax IV)

$\Rightarrow X' \vdash \neg \forall x\, \alpha$; $X' \vdash \forall x\, \alpha$ (MP)

$\Rightarrow X' \vdash \bot$

#

$\square$

$\boxed{\text{Henkin Set}}$

$X \subseteq \mathcal{L}$ is a Henkin Set
if X satisfies the following two conditions

(H1) $X \vdash \neg\alpha \iff X \nvdash \alpha$

(H2) $X \vdash \forall x\, \alpha \iff X \vdash \alpha^{x}_{c}$
for all constant $c \in \mathcal{L}$ □

(H3) For each term $t \in \mathcal{L}$ there is a constant $c \in \mathcal{L}$ such that
$X \vdash t = c$.

(H1) $\wedge$ (H2) $\Rightarrow$ (H3)

$\boxed{\text{Lemma H}}$ Let $X \subseteq \mathcal{L}$ be consistent (i.e. $X \nvdash \perp$).
Then there exists a Henkin Set $Y \supseteq X$ in a suitable constant expansion $\mathcal{L}C$ of $\mathcal{L}$. □

(L1) Every Henkin set has a model (TERM MODEL).
(L2) Every consistent set has a model.
(L3) Every consistent set is satisfiable.

$\Rightarrow$

$\boxed{\text{COMPLETENESS THEOREM.}}$

Let $\mathcal{L}$ denote any first order language. $X \subseteq \mathcal{L}$, $\alpha \in \mathcal{L}$.
Then
$X \vdash \alpha \iff X \vDash \alpha$, for all $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$.
□

Proof of LEMMA H.

$\mathcal{L}_0 := \mathcal{L}$, $X_0 := X$;

Assume that we have already defined

$\mathcal{L}_n$ & X_n.

Take all $\alpha \in \mathcal{L}_n$ and $x \in var$.

Construct new constants...

$$c_{x,\alpha,n} := c_n$$
$$\mathcal{L}_{n+1} := \mathcal{L}_n c_n$$
$$\Gamma_{\mathcal{L}_n} := \{\neg \alpha^x \mid \alpha \in \mathcal{L}_n, x \in var\}$$
$$X_{n+1} := X_n \cup \Gamma_{\mathcal{L}_n}$$

$X_{n+1} \subseteq \mathcal{L}_{n+1}$ & X_{n+1} = consistent. (Lemma pp 63)

$\therefore$

$\forall_n \; X_n \nvdash \bot$, $X_n \subseteq \mathcal{L}_n$. (By induction)

$$X' := \bigcup_{n \in \mathbb{N}} X_n \; ; \quad \mathcal{L}' := \bigcup_{n \in \mathbb{N}} \mathcal{L}_n \; ; \quad C := \bigcup_{n \in \mathbb{N}} c_n .$$

(A) $X' \nvdash \bot$ (By Finitenss of proof)

(B) $\forall_{\alpha \in \mathcal{L}', x \in var}$ $\neg \alpha^x \in X'$

$\alpha \in \mathcal{L}', x \in var \Rightarrow \alpha \in \mathcal{L}_n$ (for minimal such n), $x \in var$

$\Rightarrow \neg \alpha^x \in X_{n+1}$

$\Rightarrow \neg \alpha^x \in X'$

(C) X' has a maximal consistent extension Y s.t.

$\forall_{\alpha \in \mathcal{L}', x \in var}$ $Y \vdash \neg \alpha^x$

Let $(H, \subseteq)$ be the partial order over all consistent extensions of X' in $\mathcal{L}'$. $H \neq \phi$, since $X' \in H$.

Every chain $K \subseteq H$ has an upper bound in H

$= \bigcup K$, $\bigcup K \nvdash \bot$ ($\because$ Every member is consistent.)

By Zorn's Lemma, H has a consistent maximal element

$Y \supseteq X'$. $Y \nvdash \bot$. $\boxed{Y = \text{Henkin}}$

(D) $Y \vdash \neg\alpha^{x}$ $\quad \forall_{\alpha \in \mathcal{L}',\, x \in \mathrm{var}}$.

$Y \supseteq X'$ and $\neg\alpha^{x} \in X'$ (by construction)

(H1) $Y \vdash \alpha \Leftrightarrow Y \vdash \neg\alpha$ $\quad$ ($\because$ $Y =$ consistent)

(H2) $(\Rightarrow)$

$$\dfrac{Y \vdash \forall x\, \alpha\,, \quad Y \vdash \forall x\, \alpha \to \alpha^{x}_{c} \quad (\mathrm{Ax\,II})}{Y \vdash \alpha^{x}_{c}}\,(\mathrm{MP})$$

$(\Leftarrow)$ $\quad Y \vdash \alpha^{x}_{c} \quad \forall_{c \in \mathcal{L}'}$ $\quad$ (i.e. $Y \vdash \alpha^{x}_{c}$, $c := c_{x,\alpha,n}$ $\alpha \in \mathcal{L}_n$)

Suppose $Y \nvdash \forall x\, \alpha$

$$\Rightarrow \dfrac{\dfrac{\dfrac{Y \vdash \neg\forall x\, \alpha \text{ (By H1)}; \quad Y \vdash \alpha^{x}_{c}}{Y \vdash \neg\forall x\, \alpha \wedge \alpha^{x}_{c}}}{Y \vdash \alpha^{x}} \qquad Y \vdash \neg\alpha^{x} \quad (\because \neg\alpha^{x} \in \Gamma_{\mathcal{L}_n} \subseteq Y)}{Y \vdash \bot \;\Rightarrow\; \#.}$$

$\square$

Consider a term t and a new variable $x \notin \mathrm{var}\, t$.

$$\alpha := t \neq x$$

$Y \vdash \neg\forall_x\; t \neq x$

$\Rightarrow Y \nvdash \forall x\; t \neq x$ $\quad$ (By H1)

$\Rightarrow Y \nvdash t \neq c$ $\quad$ for some c $\quad$ (By H2)

$\Rightarrow Y \vdash t = c$ $\quad$ for some c $\quad$ (By H1).

Thus. (H3) For each term t there is a constant c s.t

$$Y \vdash t = c.$$

$\square$

$\boxed{\text{TERM MODEL}}$

T = Term algebra of all the terms in $\mathcal{L}$.
Y = Henkin Set $Y \subseteq \mathcal{L}$.
Equivalence Relation on the Set T: $\approx$

$$t \approx t' \quad \text{iff} \quad Y \vdash t = t'$$

Thus for every variable $x \in var$ there is a constant c s.t

$$x \approx c \quad \text{and} \quad Y \vdash x = c.$$

$\Rightarrow$ ρ = Variable assignment.

$A := T/\approx$ $\left\{ \begin{array}{l} \text{Partition of } T \text{ into equivalence classes} \\ \text{with respect to the relation } \approx \end{array} \right.$

Model: $M := (A, \rho)$ $\leftarrow$ Show that M is a model of Y.

$\boxed{\text{MAIN LEMMA}}$ Every Henkin Set $Y \subseteq \mathcal{L}$ possesses a (term) model.

proof: Construct a term model $M := (A, \rho)$ as above.

(I) $\approx$ is a congruence relation
For every n-ary function f, and n-ary relation P, and terms t_i and t_i'

$$t_1 \approx t_1' \wedge t_2 \approx t_2' \wedge \cdots \wedge t_n \approx t_n'$$
$$\Rightarrow f\vec{t} \approx f\vec{t}\,' \wedge P\vec{t} \approx P\vec{t}\,'$$

(II) For each term $t \in T$, there is a constant c s.t. $c \approx t$
($\because$ Y = Henkin, use (H3)).

$$M := (A, \rho)$$
$$A := \{\bar{c} \mid c \in C\}; \quad x^M := \bar{x}; \quad c^M := \bar{c};$$
$$f^M(\bar{t}_1, \ldots, \bar{t}_n) := \overline{f(t_1, \ldots, t_n)}$$
$$P^M(\bar{t}_1, \ldots, \bar{t}_n) := \overline{P(t_1, \ldots, t_n)} \Leftrightarrow Y \vdash P(t_1, \ldots, t_n)$$

We need to prove

(A) $t^{\mathcal{M}} = \bar{t}$

(B) $\mathcal{M} \models \alpha$ iff $Y \vdash \alpha$

(A) By induction on the structure of the term $\bar{t}$.

$$t_i^{\mathcal{M}} = \bar{t}_i \qquad i = 1, \dots, n$$
$$t^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}}) = f^{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_n) = \overline{f(t_1, \dots, t_n)} = \bar{t}$$

(B) By induction on structure of α

$$\mathcal{M} \models t = s \iff t^{\mathcal{M}} = s^{\mathcal{M}} \iff \bar{t} = \bar{s} \iff t \approx s \iff Y \vdash t = s$$

$$\mathcal{M} \models P\vec{t} \iff P^{\mathcal{M}} t_1^{\mathcal{M}} \dots t_n^{\mathcal{M}} \iff P^{\mathcal{M}} \bar{t}_1 \dots \bar{t}_n \iff Y \vdash P\vec{t}$$

$$\mathcal{M} \models \neg\alpha \iff \mathcal{M} \not\models \alpha \iff Y \nvdash \alpha \ (IH) \iff Y \vdash \neg\alpha \quad (H1)$$

$$\mathcal{M} \models \alpha \wedge \beta \iff \mathcal{M} \models \alpha \text{ and } \mathcal{M} \models \beta \iff Y \vdash \alpha \text{ and } Y \vdash \beta \ (IH) \iff Y \vdash \alpha \wedge \beta \quad (\wedge 1 \,\&\, \wedge 2)$$

$$\mathcal{M} \models \forall x\, \alpha \iff \mathcal{M}_{\bar{c}}^{x} \models \alpha \text{ for all } c \in C \iff \mathcal{M}_{c^{\mathcal{M}}}^{x} \models \alpha \iff \mathcal{M} \models \alpha_c^x \quad \forall c \in C$$
$$\iff Y \vdash \alpha_c^x \iff Y \vdash \forall x\, \alpha \quad (H2) \qquad \square$$

Corollary 1: $\boxed{\text{Model Existence Thm:}}$ $\left\{ \begin{array}{l} \text{Each consistent } X \subseteq \mathcal{L} \\ \text{has a model.} \end{array} \right.$

Corollary 2: $\boxed{\text{Completeness Thm}}$