November 5 2013

Lecture #8

Proof System for First-Order Logic.

Axioms Λ

(I) Tautologies:

(II) $\forall x\, \alpha \to \alpha^x_t$, where t is substituable for x in α.

(III) $\forall x\, (\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$

(IV) $\alpha \to \forall x\, \alpha$

(V) $x = x$

(VI) $x = y \to (\alpha \to \alpha')$, where α = atomic and α' = obtained from α by replacing x in zero or more places by y.

{Note: V & VI axioms assume that the underlying language includes equality.}

Rule of Inference

Modus Ponens (MP) $\dfrac{\alpha, \ \alpha \to \beta}{\beta}$

Deduction $\Gamma \vdash \varphi$ A deduction of φ from Γ is a sequence $\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$ of wff's such that

- $\alpha_n = \varphi$
- For each $i \leq n$ either
  - (i) $\alpha_i \in \Gamma \cup \Lambda$, or
  - (ii) for some $j, k < i$ α_i obtained by modus ponens from α_j & α_k $\dfrac{\alpha_j, \alpha_k}{\alpha_i}$ (MP) $\{ \alpha_k \equiv \alpha_j \to \alpha_i$ or $\alpha_j \equiv \alpha_k \to \alpha_i$ □

PROOF = A finite sequence of fixed indisputable steps

BUILT FROM

- Axioms { Self-evidently true facts. Do not need proofs.
- Rule of Inference { Derive new facts from axioms and other facts that have already been derived

PROOF = An effective mechanism to convince a skeptic.

→ Derivation is mechanically checkable.

The skeptic does not need to enumerate all possible models and variable assignments.

Example: $\vdash \forall x\,(Px \to \exists y\, Py)$

① $\forall x\,[((\forall y\, \neg Py) \to \neg Px) \to (Px \to (\neg \forall y\, \neg Py))]$ { Tautology "Contra-positivity"

② $\forall x\,(\forall y\, \neg Py \to \neg Px) \to \forall x\,(Px \to \neg\forall y\, \neg Py)$ { III + MP(1) }

③ $\forall y\, \neg Py \to \neg Px$ { II }

④ $\forall x\,(\forall y\, \neg Py \to \neg Px)$ { IV & MP(3) }

⑤ $\forall x\,(Px \to \neg\forall y\, \neg Py)$ MP(2; 4)

$\equiv \forall x\,(Px \to \exists y\, Py)$

Defn

A set of formulas Δ is <u>closed under modus ponens</u> iff
$\alpha \in \Delta$; $\alpha \to \beta \in \Delta$ implies $\beta \in \Delta$.

<u>Induction Principle</u>. S = Set of wff's
$\Gamma \cup \Lambda \subseteq S$
S: minimal set containing $\Gamma \cup \Lambda$ and closed under MP
$S = \{\alpha \mid \Gamma \vdash \alpha\}$ = Set of all theorems of Γ.

SOUNDNESS THEOREM (for 1st order Logic).

If $\Gamma \vdash \phi$ then $\Gamma \models \phi$.

Completeness
If $\Gamma \models \phi$ then $\Gamma \vdash \phi$

{Every consistent set of formulas is satisfiable.}

Proof in 3 steps.

Substitution Lemma $\mathcal{M} \models_\rho \phi^x_t$ iff $\mathcal{M} \models_{\rho(x|\bar{\rho}(t))} \phi$
Generalization Lemma $\Gamma \vdash \phi$ implies $\Gamma \vdash \forall x \phi$ (certain condns on x)
Deduction Lemma $\Gamma \cup \{\gamma\} \vdash \phi$ implies $\Gamma \vdash \gamma \to \phi$.

$\boxed{\text{SOUNDNESS THEOREM:}}$ $\Gamma \vdash \phi$ implies $\Gamma \models \phi$

Proof (By Induction)
(1) ϕ is a logical axiom. $\models \phi$; thus, $\Gamma \models \phi$ ($\because$ All axioms are sound)
(2) $\phi \in \Gamma$, Thus, $\Gamma \models \phi$ (by defn).
(3)
$$\frac{\Gamma \vdash \psi \; ; \; \Gamma \vdash \psi \to \phi}{\Gamma \vdash \phi} \text{ MP.}$$

By Ind Hyp: $\Gamma \models \psi$, $\Gamma \models \psi \to \phi \equiv \Gamma \models \neg\psi \vee \phi$
$\Gamma \models \psi \wedge (\neg\psi \vee \phi) = (\psi \wedge \neg\psi) \vee (\psi \wedge \phi) = (\psi \wedge \phi)$.
$\therefore \Gamma \models \phi$.

Remaining Task: Check that all axioms are sound.
(TRIVIAL – Except Axiom Gp I)

$\forall x\, \alpha \to \alpha^x_t$ where t is substituted for x in α.

Suppose $\mathcal{M} \models_\rho \forall_x \alpha$

Thus $\forall d \in \mathrm{dom}(\mathcal{M})$ $\mathcal{M} \models_{\rho(x|d)} \alpha$

Make $d = \bar{\rho}(t)$

Then $\mathcal{M} \models_{\rho(x|\bar{\rho}(t))} \alpha$

By substitution lemma $\mathcal{M} \models \alpha^x_t$ $\square$.

$\boxed{\text{Substitution Lemma}}$

If the term t is substituted for the variable x in a wff ϕ, then for any model $\mathcal{M}$ and variable assignment ρ

$$\mathcal{M} \models_\rho \phi^x_t \quad \text{iff} \quad \mathcal{M} \models_{\rho(x|\bar{\rho}(t))} \phi.$$

Proof: By induction on the structure of the wff ϕ. $\square$

$\boxed{\text{Generalization Lemma}}$

If $\Gamma \vdash \phi$ and x does not occur free in any formula in Γ, then $\Gamma \vdash \forall x\, \phi$.

Proof: Show that
$\Gamma \cup \Lambda \subseteq \{\phi \mid \Gamma \vdash \forall x\, \phi\} = S$ is closed under MP.

<u>CASE 1.</u> Suppose $\phi \in \Gamma$. Then x does not occur free in ϕ.
By axiom Gp IV : $\phi \to \forall x\, \phi$

$$\frac{\begin{array}{l}\Gamma \vdash \phi \\ \Gamma \vdash \phi \to \forall x\, \phi\end{array}}{\Gamma \vdash \forall x\, \phi}\ \text{MP.}$$

<u>Case 2.</u> Suppose $\phi \in \Lambda$ (tautology). $\forall x\, \phi$ ≡ Also a tautology
≡ Axiom:

$\Gamma \vdash \forall x\, \phi$.

<u>Case 3.</u>

Suppose ϕ is derived by MP.

$$\frac{\Gamma \vdash \psi\ ;\ \Gamma \vdash \psi \to \phi}{\Gamma \vdash \phi}$$

Inductive Hyp. $\Gamma \vdash \forall x\, \psi$ (a) $\Gamma \vdash \forall x\,(\psi \to \phi)$. (b)

But by axiom Gp III $\Gamma \vdash \forall x(\psi \to \phi) \to (\forall x \psi \to \forall x \phi)$ (c)

$$\frac{(a) \quad \dfrac{(b) \quad (c)}{\Gamma \vdash \forall x\, \psi \to \forall x\, \phi}\ \text{MP}}{\Gamma \vdash \forall x \phi.}\ \text{MP}$$

□

DEDUCTION LEMMA If $\Gamma \cup \{\gamma\} \vdash \phi$ then $\Gamma \vdash \gamma \to \phi$

Proof: Defn: A tautologically implies B ≡ Whenever A is true, B must be true.

$\Gamma \cup \{\gamma\} \vdash \phi$
iff $\Gamma \cup \{\gamma\} \cup \Lambda$ taut. implies ϕ
iff $\Gamma \cup \Lambda$ taut. implies $\gamma \to \phi$
iff $\Gamma \vdash \gamma \to \phi$.

□