Sept 17, 2013

Lecture #3

PL = Propositional Logic:

Boolean Connectives
E.g.
- AND $\wedge$
- OR $\vee$
- IMPLY $\Rightarrow$
- NEGATION $\neg$

All can be derived
NAND, $\uparrow$,
"Incompatibility"
$\neg(A \wedge B) \equiv A \uparrow B$
A is incompatible with B.

① $A \wedge B$ is true iff A and B are both true; and false, otherwise.

$$\wedge : \{0,1\}^2 \to \{0,1\}$$
$$: (1,1) \mapsto 1 \qquad : (1,0) \mapsto 0$$
$$: (0,1) \mapsto 0 \qquad : (0,0) \mapsto 0$$

$$\text{graph } \wedge = \{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$$

$$A \cdot B \equiv \text{multiplication over } \mathbb{Z}_2 = \{0,1\}$$

Value Matrix: $\quad \circ : \{0,1\}^2 \to \{0,1\}$

$$\begin{pmatrix} 1\circ 1 & 1\circ 0 \\ 0\circ 1 & 0\circ 0 \end{pmatrix}$$

Truth Table.

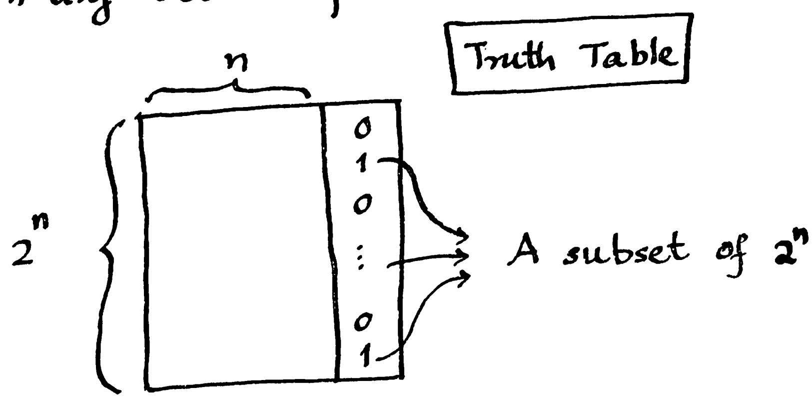
Truth Table / Value Matrix for $\wedge$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Boolean / Truth Function

A function $f : \{0,1\}^n \to \{0,1\}$ is called an n-ary Boolean function or truth function.

$\mathbb{B}^n$ = n-ary Boolean function



$$|\mathbb{B}^n| = 2^{2^n}$$

$\mathbb{B}^2$ = 2-ary Boolean functions. $\quad |\mathbb{B}^2| = 2^{2^2} = 16$

Following is a list of such Boolean functions:

(1) Conjunction (And)
A and B $\quad A \wedge B$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(2) Disjunction (Inclusive Or)
A or B $\quad A \vee B$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(3) Implication
If A then B $\quad A \Rightarrow B \quad$ { Equivalent to $\neg A \vee B \equiv \neg(A \wedge \neg B)$ }
(B provided A)

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

4. Equivalence

A iff B $\quad A \Leftrightarrow B$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Exclusive Disjunction (Parity)

A xor B $\quad (A+B) \bmod 2$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6. Nihilation

A nor B $\quad A \downarrow B$

(neither A nor B)

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

7. Incompatibility

A nand B $\quad A \uparrow B$

(Not at once A and B)

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

+ Some trivial ones; $\perp$, A, B, $\neg A$, $\neg B$, T

$B \Rightarrow A$, $A \not\Rightarrow B$, $B \not\Rightarrow A$;

FORMALISM.

A formal language.

Propositional formula: Strings of symbols (over an alphabet $A \equiv PV \cup LC$) built in given ways from basic symbols.

Parantheses $\equiv \{ (,) \}$

PV $\equiv$ Propositional Variables; Symbolized by $p_0, p_1, p_2, \ldots$ etc

LC $\equiv$ Logical Connectives ; Symbolized by $\wedge, \vee, \neg, \Rightarrow, \ldots$

$$(p_1 \vee \neg p_2 \vee p_3) \wedge (p_2 \vee \neg p_3 \vee p_4) \wedge (p_4 \vee \neg p_5 \vee \neg p_6)$$

Well-formed formulas (wff's).

$$\mathcal{F} ::= p_i \mid (\mathcal{F}_1 \wedge \mathcal{F}_2) \mid (\mathcal{F}_1 \vee \mathcal{F}_2) \mid \cdots \mid \neg \mathcal{F}$$

For example:

$(p_1 \wedge (p_2 \vee \neg p_1))$ = Valid wff.

Propositional Language:

$\mathcal{F}$ of formulas built up from the symbols (logical signature)

$($, $)$, $\wedge$, $\vee$, $\neg$, ... and

logical variables

$p_0, p_1, p_2, \ldots$

inductively as follows:

(F_1) The atomic strings $p_0, p_1, p_2, \ldots$ are formulas, called <u>prime formulas</u> (also called <u>atomic formulas</u> or just <u>primes</u>)

(F_2) If the strings α and β are formulas, then so too are strings

$$(\alpha \wedge \beta),\quad (\alpha \vee \beta) \text{ and } \neg\alpha.$$

Set-Theoretic Statement:

$\mathcal{F}$ = the smallest (i.e. the intersection) of all sets of strings S built from the logical signatures and propositional variable symbols with the properties:

(F_1) $p_0, p_1, p_2, \ldots \in S$

(F_2) $\alpha, \beta \in S \Rightarrow (\alpha \wedge \beta), (\alpha \vee \beta), \neg\alpha \in S$

$$\mathcal{F} = \bigcap S.$$

$$\begin{tikzcd}[column sep=small] & \top \arrow[dl, dash] \arrow[dr, dash] & \\ T \arrow[dr, dash] & & F \arrow[dl, dash] \\ & \bot & \end{tikzcd}$$

Lattice

T: true , F: False

$$\top \equiv (T \vee F) \equiv (\alpha \vee \neg\alpha)$$
$$\bot \equiv (T \wedge F) \equiv (\alpha \wedge \neg\alpha) \equiv \neg\top$$

Always True $\to$ Tautology, Verum, Top, $\top$.

Always False $\to$ Contradiction, Falsum, Bottom, $\bot$.

Note: We are using LEM ($\equiv$ Law of Excluded Middle)

Our logic is 2-valued,
A proposition is either true or false.

Boolean Formulas.

Obtained using only three Boolean connectives;

$$\{\wedge, \vee, \neg\}$$

Other connectives

$$\alpha \Rightarrow \beta \equiv \neg(\alpha \wedge \neg\beta) \equiv \neg\alpha \vee \beta$$

$$\alpha \Leftrightarrow \beta \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$$
$$\equiv ((\neg\alpha \vee \beta) \wedge (\neg\beta \vee \alpha))$$
$$\equiv (\neg\alpha \wedge \neg\beta) \vee (\alpha \wedge \beta)$$

<u>Notations</u>:

$p, q, \dots \equiv$ PV, propositional variable

$\alpha, \beta, \dots \equiv \mathcal{F}$, formulas (wff)

$\pi \equiv \mathcal{PF}$, prime formulas.

$X, Y, Z, \dots \equiv$ PF, Propositional formulas.

INDUCTION

(on the construction of a formula, i.e., its parse-tree.)

PRINCIPLE OF FORMULA INDUCTION:

$$\frac{\mathcal{E}\pi;\quad \mathcal{E}\alpha, \mathcal{E}\beta \Rightarrow \mathcal{E}(\alpha\wedge\beta),\ \mathcal{E}(\alpha\vee\beta),\ \mathcal{E}(\neg\alpha)}{\mathcal{E}\varphi \quad (\forall\varphi\in\mathcal{F})}$$

Let $\mathcal{E}$ be a property of strings that satisfy the following conditions:

(0) Base Case : $\mathcal{E}\pi$ holds for all prime formulas π.

(S) Induction Case:
For all $\alpha, \beta \in \mathcal{F}$, the following holds:

$$\mathcal{E}\alpha,\ \mathcal{E}\beta \Rightarrow \mathcal{E}(\alpha\wedge\beta),\ \mathcal{E}(\alpha\vee\beta),\ \mathcal{E}(\neg\alpha)$$

Then $\mathcal{E}\varphi$ holds for all formulas (wff's) φ.

Examples.

Ex 1. Rank:

$\mathrm{rk}\,\varphi \equiv$ highest number of nested connections in φ.

$\pi \in \mathcal{PF}$ $\quad \mathrm{rk}\,\pi \equiv 0$

$\mathrm{rk}\,\neg\alpha \equiv 1 + \mathrm{rk}\,\alpha$

$\mathrm{rk}\,(\alpha \circ \beta) \equiv 1 + \max(\mathrm{rk}\,\alpha, \mathrm{rk}\,\beta)$ $\quad \circ \in \{\wedge, \vee\}$

Binary Connectives.

Ex 2. Length

$\ln \varphi =$ number of ~~str~~ characters in φ

$\pi \in \mathcal{PF}$ $\quad \ln \pi \equiv 1.$

$\ln \neg\alpha \equiv 1 + \ln \alpha$

$\ln(\alpha \circ \beta) \equiv 1 + \ln \alpha + \ln \beta.$

Ex 3 Subformulas.

$SF\,\varphi =$ Subformulas of φ

$\pi \in \mathcal{PF}$ $\quad SF\,\pi \equiv \{\pi\}$

$SF\,\neg\alpha \equiv SF\,\alpha \cup \{\neg\alpha\}$

$SF\,(\alpha \circ \beta) \equiv \cancel{SF(\alpha) \cup SF(\beta)}$

$SF\alpha \cup SF\,\beta \cup \{\alpha \circ \beta\}.$

THM. $\quad |SF\varphi| \leq \ln \varphi.$

Proof

$\pi \in \mathcal{PF}$ $\quad |\{\pi\}| = 1 \leq 1.$

$$|SF\,\neg\alpha| \equiv |SF\,\alpha \cup \{\neg\alpha\}| = 1 + |SF\alpha| \leq 1 + \ln \alpha = \ln \neg\alpha$$

$$\begin{aligned} |SF\,(\alpha \circ \beta)| \equiv |SF\,\alpha \cup SF\,\beta \cup \{\alpha \circ \beta\}| &\leq 1 + |SF\,\alpha| + |SF\,\beta| \\ &\leq 1 + \ln \alpha + \ln \beta \\ &= \ln(\alpha \circ \beta) \end{aligned}$$

TRUTH VALUE

Truth value of a connected sentence depends only on the truth values of its constituent parts.

$$\omega : PV \to \{0,1\}$$

Extend the mapping to all wff.

$$\omega : \mathcal{F} \to \{0,1\}$$

$\pi \in PF$ $\qquad \omega(\pi) \equiv$ defined by $\omega : PV \to \{0,1\}$

$$\omega \neg\alpha \equiv 1 - \omega\alpha$$
$$\omega(\alpha \wedge \beta) \equiv \omega\alpha \cdot \omega\beta$$
$$\omega(\alpha \vee \beta) \equiv \max(\omega\alpha, \omega\beta)$$

Note

$$\omega T = \omega(\alpha \vee \neg\alpha) = \max(\omega\alpha, \omega\neg\alpha)$$
$$= \max(\omega\alpha, 1 - \omega\alpha) = 1$$

$$\omega \perp = \omega(\alpha \wedge \neg\alpha) = \omega\alpha \cdot \omega\neg\alpha$$
$$= \omega\alpha(1 - \omega\alpha)$$
$$= \omega\alpha - \omega\alpha \cdot \omega\alpha = \omega\alpha - \omega\alpha \equiv 0.$$