December 3 2013

$\boxed{\text{LECTURE \# 12}}$

PA = Peano Arithmetic. $\qquad \mathcal{L}_{AR} = (0, S, +, \times)$

Axioms
$$\begin{cases} \forall x \quad 0 \neq Sx \\ \forall xy \quad Sx = Sy \rightarrow x = y \\ \forall x \quad x + 0 = x \,; \quad \forall x, y \quad x + Sy = S(x+y) \\ \forall x \quad x \times 0 = 0 \,; \quad \forall x, y \quad x \times Sy = x \times y + x \end{cases}$$

Induction Axiom/Schema
$$\forall y \left[\{\varphi(0, y) \wedge \forall x \; \varphi(x, y) \rightarrow \varphi(Sx, y)\} \rightarrow \forall x \; \varphi(x, y) \right]$$

$\boxed{\text{Ordinals.}}$

Natural Numbers: $\mathbb{N} \quad \begin{cases} 0 \in \mathbb{N} \\ \forall_{n \in \mathbb{N}} \; S(n) \in \mathbb{N} \end{cases}$ $\Big\}$ An injective successor function S.

ORD : A sequential compactification of the set $\mathbb{N}$.

1) $0 \in$ ORD
2) $\alpha \in$ ORD $\rightarrow S(\alpha) \in$ ORD $\qquad$ Successor Ordinal
3) $\{\alpha_i\}_{i \in \mathbb{N}} \subseteq$ ORD $\rightarrow \alpha = \lim\limits_{j \in \mathbb{N}} \alpha_{i_j}$ $\qquad$ Limit Ordinal

[Whenever $\alpha_i \in$ ORD is a sequence of ordinals indexed by natural numbers, there exists $\alpha \in$ ORD such that $\alpha_{i_j} \rightarrow \alpha$ for some subsequence i_j.]

$\boxed{\text{The class ORD is well-ordered:}}$

a) $\alpha \in$ ORD $\rightarrow \alpha \not< \alpha$; b) $\alpha, \beta \in$ ORD $\rightarrow \alpha < \beta$ or $\beta < \alpha$

c) $\alpha, \beta, \gamma \in$ ORD $\quad \alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma$

d) ORD does not have an infinite descending chain
$$\alpha_1 > \alpha_2 > \alpha_3 > \cdots$$

AC ≡ $\boxed{\text{Well-Ordering theorem:}}$

Every set can be well-ordered. □

ORD: $0, 1, 2, \ldots$

$\omega = \text{lub}\{0, 1, 2, 3, \ldots\}$

↳ least upper bound

$\omega+1, \omega+2, \ldots$

$\omega\cdot 2 = \text{lub}\{\omega, \omega+1, \omega+2, \ldots\}$

$\omega\cdot 3, \omega\cdot 4, \ldots$

$\omega^2 = \text{lub}\{\omega, \omega\cdot 2, \omega\cdot 3, \ldots\}$

$\omega^3, \omega^4, \ldots$

$\omega^\omega = \text{lub}\{\omega, \omega^2, \omega^3, \omega^4, \ldots\}$

$\vdots$

$\omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$

$\epsilon_0 = \text{lub}\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} \equiv \omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}$

Note $\boxed{\epsilon_0 = \omega^{\epsilon_0}}$

<u>HORD</u>: Harmless Ordinals:

Every $n \in \mathbb{N}$ is a harmless ordinal.

For $i = 1, \ldots, t$, let $n_i \in \mathbb{N}\setminus\{0\}$ and $\alpha_i \in \text{HORD}$

$\alpha_i > \alpha_{i+1}$

Then

$$\alpha = n_1\omega^{\alpha_1} + n_2\omega^{\alpha_2} + \cdots + n_t\omega^{\alpha_t} \in \text{HORD}$$

is a harmless ordinal.

1) The order of harmless ordinals with the same exponents is just the lexicographic order.

2) α = limit ordinal if $\alpha_t > 0$.

3) $\gamma_0 = 1$ and $\gamma_{n+1} = \omega^{\gamma_n} \Rightarrow \epsilon_0 = \lim_n \gamma_n \notin \text{HORD}$.

= Smallest 'Harmful' ordinal.

= HORD.

Let $\alpha \in$ HORD

$$\alpha = n_1 \omega^{\alpha_1} + n_2 \omega^{\alpha_2} + \cdots + n_t \omega^{\alpha_t}$$

$T(\alpha) = t$; $N(\alpha) = \max\{n_1, \ldots, n_t, N(\alpha_1), \ldots, N(\alpha_t)\} + 1.$

RAPIDLY GROWING FUNCTIONS

For $\alpha \in$ HORD

$$\begin{cases} f_0(n) := Sn \\ f_\alpha(n) := f_\beta^{n}(n) & \text{if } \alpha = S\beta \\ f_\alpha(n) = f_{\alpha(n)}(n) & \text{if } \alpha = \mathrm{lub}\{\alpha(0), \alpha(1), \ldots, \alpha(n), \ldots\} \end{cases}$$

Also define: $f_{\epsilon_0}(n) = f_{r(n)}(n)$.

<u>Note</u>:

$$f_0(n) = n+1$$

$$f_1(n) = n + \underbrace{1 + 1 \cdots 1}_{n} = 2n$$

$$f_2(n) = \underbrace{2 \times 2 \times \cdots \times 2n}_{n} = 2^n n$$

$$f_3(n) \approx 2^{\cdot^{\cdot^{2}}}\}n = 2\uparrow n$$

$$f_4(n) \approx 2\underbrace{\uparrow\uparrow\cdots\uparrow}_{n} n = 2\uparrow\uparrow n$$

$$\vdots$$

$$f_\omega(n) = \text{Ackerman's Function}$$

$$\vdots$$

$\boxed{\text{GENTZEN}}$ (1950) Let $\phi(n,k)$ be a binary predicate such that

$PA \vdash \forall_n \exists_k : \phi(n,k)$

Let $f_\phi(n) = \min_k \phi(n,k)$

Then there exists an $\alpha \in$ HORD s.t.

$f_\phi < f_\alpha$ $\square$ $\{f_\phi$ grows slower than $f_\alpha\}$

- We want to discover a "true" statement using a predicate $P(s, m)$, such that its associated function

$$\varphi : \mathbb{N} \to \mathbb{N}$$
$$: s \mapsto m$$

satisfying

$$\forall s \, [P(s, \varphi(s))] \quad \text{holds true}$$

But

$$\varphi \approx f_{\epsilon_0}.$$

Where do we find such a natural statement?

> MOTZKIN'S RULE:
> "Complete disorder is impossible."

- For every s, if you want to find a "particular" structure of size s, look for it in a superstructure of size m.

And you will always succeed; however, m may be UNIMAGINABLY LARGE.

How can you ~~we~~ show ~~prove~~ Motzkin's Rule to be TRUE?

a) Start with an infinitary version: If you want to find a particular infinitary structure, look for it in an infinitary ~~st~~ superstructure.

b) Use compactness: König's Lemma.

$$\Downarrow$$

(I) RAMSEY'S THEOREM

(II) PARIS-HARRINGTON THEOREM (generalizes Ramsey)

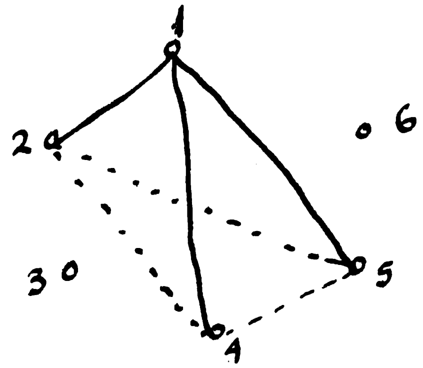
(III) KÖNIG'S LEMMA.

$\boxed{\text{Ramsey's Theorem:}}$

order { Suppose you wish to find
(a) Three mutual friends
or (b) Three mutual un-friends

Disorder { Any "random" group of six-people will suffice.
(Social-Network)

Take a complete graph $K_6 = (V, E)$ with its edges colored in two colors: $c: E \to \{0, 1\}$. Then the graph $\langle K_6, c \rangle$, for all c, must have a monochromatic triangle.



1) Each vertex in K_6 has five neighbors. E.g. vertex v_1.
2) In any two-colored K_6, at least 3 of v_1's edges must have the same color (say, black, wlog)
3) For this $\langle K_6, c \rangle$ to avoid black-triangles, it must be the case that ends of every pair of v_1's black edges must be colored gray.
4) But then $\langle K_6, c \rangle$ contains a gray triangle.
{ Three of v_1's friends are mutual unfriends. }

<u>Theorem</u>: For every $n \geq 6$, any 2-colored K_n contains a monochromatic K_3.

$\rightarrow$ Predicate $\cancel{R(3,6)}$ $R(2,3,6)$ $\qquad$ $\cancel{PA \vdash R(3,6)}$. $PA \vdash R(2,3,6)$

A generalization:

$\boxed{\text{Ramsey Theorem}}$

For any $k, m \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, a k-colored K_n contains a monochromatic K_m.

$$\forall_{k,m} \exists_N \; R(k,m,N) \quad \square$$

A hyper-graph version:

$\boxed{\text{Ramsey Theorem (Finite)}}$ $R_F T$

For any $i, k, m \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that if the i-element subsets of a set with N or more elements are colored in k colors, then there is a subset of size m whose i-elements subsets all have the same color.

$$\forall_{i,k,m} \exists_N \; R(i,k,m,N) \quad \square$$

$\boxed{\text{PARIS - HARRINGTON THEOREM}}$ PH.

Let $h : \mathbb{N} \to \mathbb{N}$ be increasing.
A set $X \subseteq \mathbb{N}$ is h-large if $|X| \geq h(\min X)$.

$X = \{3, 5, 26, 1767\}$: S-large $\quad |X| = 4 \geq S \min X = S3$.
$X = \{5, 26, 1767, 10^{10}\}$ = S-small $\quad |X| = 4 < S5 = 6$

For all $i, k, m, N \in \mathbb{N}$, we denote

$$N \xrightarrow{h} (m)^i_k \quad \equiv \quad PH_h(i,k,m,N)$$

if for all k-colorings of $[N]^i \equiv$ i-subsets of $\{0, \dots, N\}$ there exists an h-large monochromatic subset of $[N]$ of size at least m.

$\cancel{\mathbb{N} \nvdash PH(i,k,m,N)}$ For all $h : \mathbb{N} \to \mathbb{N}$ increasing,

$$\forall_{i,k,m} \exists_N \; PH_h(i,k,m,N) \quad \square$$

All $R_F T$'s and PH follow from $R_I T$

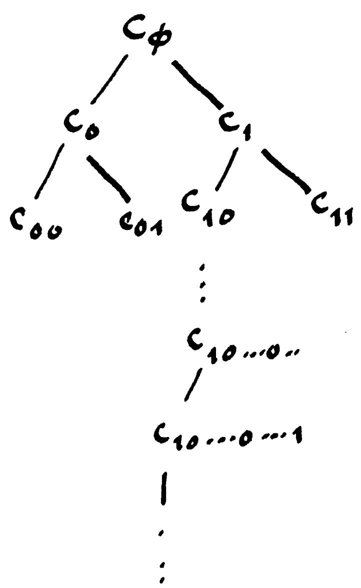
$\boxed{\text{Infinite Ramsey Theorem}}$ (Ramsey 1930)

For any i, k and any k-coloring of the i-element subsets of a countably infinite set (e.g. $\mathbf{N}$), there is an infinite subset whose i-element subsets all have the same color. □

proof sketch: $i = 2$, $k = 2$

A = Countably infinite set.

For each $s \in \{0,1\}^*$, define $C_s \subseteq A$, recursively.



$C_\phi = A$

if $C_s \neq \phi$ then

$x_s \in C_s$ (arbitrary)

$C_{s|0} = \{ y \in C_s \mid y \neq x_s \wedge \{x_s, y\} = \text{gray} \}$

$C_{s|1} = \{ y \in C_s \mid y \neq x_s \wedge \{x_s, y\} = \text{black} \}$

For all n,

$$A = \{ x_t \mid t \in \{0,1\}^* \wedge \text{length}(t) \leq n \} \cup \bigcup_{\text{length}(s) = n} C_s$$

$\Rightarrow \exists_{s \in \{0,1\}^n}\ C_s$ = countably infinite.

$\therefore\ w \in \{0,1\}^\omega$ s.t. $C_{w \upharpoonright n}$ = countably infinite. [König's Lemma]

$B_0 = \{ x_{w \upharpoonright n} \mid \omega(n) = 0 \}$ and $B_1 = \{ x_{w \upharpoonright n} \mid \omega(n) = 1 \}$

$B_0 \cup B_1$ = countably infinite

$\Rightarrow$ B_0 or B_1 = countably infinite & monochromatic (by construction) for two coloring.

□

$\boxed{\text{Corollary 1: } R_F T = \text{true}.}$

proof: Suppose not. $\exists_{i,k,m} \forall_n\ R(i,k,m,n) = \text{false}$

Violates compactness.

□

However: $R_F T$ = True & $PA \vdash R_F T$.

$\boxed{\text{Corollary 2:}}$
For all $h : \mathbf{N} \to \mathbf{N}$ increasing, ~~PA(h)~~ PH_h = true. □

But for sufficiently rapidly growing h, PH_h can define a function

$$f_{PH_h} \gtrsim f_{\epsilon_0}.$$

$\therefore$ $\exists h : \mathbf{N} \to \mathbf{N}$ increasing PH_h = true, but $PA \nvdash PH_h$.

$\boxed{\text{CAVEAT}}$ We must choose a sufficiently slowly growing h, so that PH_h can be expressed in $\mathcal{L}_{AR}$.
That is, we want

$$h < f_\alpha, \quad \alpha \in \text{HORD}.$$

PARIS HARRINGTON THM:

Define $e_1(n) = n$; $e_{s+1}(n) = n^{e_s(n)}$; $h_s(n) = e_{s-1}(n) + s - 1$.
For all $s \in \mathbf{N}$, $h_s(n)$ can be defined in PA, and can be proven in PA to be total functions.

Let $PH(s, N)$ be the predicate

$$N \xrightarrow{h_{s+1}} (s)^{s+2}_{2s+1}.$$

$$f_{PH} \geq f_{\epsilon_0}.$$

$\therefore$ Sentence $S_{PH} = \forall_s \exists_N \, PH(s, N)$ is true but not provable. □

We wish to show:

$$PH(s,N) = \text{false}, \quad \text{whenever} \quad N \le f_{r_s}(s) : \begin{cases} r_0 = 1 \\ r_1 = \omega \\ r_2 = \omega^{\omega} \\ \vdots \end{cases}$$

⇒ A counter example to

$$f_{r_s}(s) \xrightarrow{h_{s+1}} (s)^{s+2}_{2s+1}$$

⇒ Show how to color the $(s+2)$-subsets of $[s, f_{r_s}(s)]$ with $2s+1$ colors so that there is no monochromatic subset X of size at least $h_{s+1}(\min X)$.

We will create an ordinal translation function: τ.

$$\tau : \mathbb{N} \to \text{HORD} \qquad \left\{ \boxed{\tau \text{ will depend on } s, r_{s+1}} \right.$$

s.t. $m, n \in \mathbb{N}$, $m \ge n \Rightarrow \tau(n) \le \tau(m)$.

⇒ Given a coloring $\chi_{s+1} : [r_{s+1}]^{s+2} \to \{0, 1, \ldots, 2s\}$ we can get a coloring

$$\chi'_{s+1} : [s, f_{r_s}(s)] \to \{0, 1, \ldots, 2s\}$$

by making

$$\chi'_{s+1}(\{n_1, \ldots, n_{s+2}\}_<) = \chi_{s+1}(\{\tau(n_1), \ldots, \tau(n_{s+2})\}_>)$$

⇒ It suffices to show that

$\boxed{\text{LEMMA}}$

For all $s \in \mathbb{N}$, there is a coloring

$$\chi_s : [r_s]^{s+1} \to \{0, \ldots, 2s-2\}$$

such that if $S \subset [r_s]$ is monochromatic for χ_s, then

$$|S| \le h_s(N(\max S)).$$

1) $\alpha \in \mathrm{HORD} \longrightarrow \quad \alpha = n_1\omega^{\alpha_1} + \cdots + n_t\omega^{\alpha_t}\,, \quad \alpha_i > \alpha_{i+1}\,.$

$T(\alpha) = t\,, \quad N(\alpha) = \max\{n_1, n_2, \dots, n_t, N(\alpha_1), \dots, N(\alpha_t)\}+1.$

2) $e_1(n) = n,\ e_{s+1}(n) = n^{e_s(n)}, \quad h_s(n) = e_{s-1}(n) + s - 1.$

3) $\gamma_0 = 1\,, \quad \gamma_{n+1} = \omega^{\gamma_n} \qquad \epsilon_0 = \lim_n \gamma_n\,.$

$\boxed{\text{Lemma}}$

Let $s > 0$; $\alpha < \gamma_{s+1}$. Then $T(\alpha) \le e_s(N(\alpha))$. □

$$\alpha = n_1\omega^{\alpha_1} + \cdots + n_t\omega^{\alpha_t}$$
$$\beta = m_1\omega^{\beta_1} + \cdots + m_u\omega^{\beta_u}$$

$\alpha > \beta \;\Rightarrow\; \alpha_1 = \beta_1,\ n_1 = m_1;\ \cdots;\ \alpha_{k-1} = \beta_{k-1},\ n_{k-1} = m_{k-1};$

$\&\,(\alpha_k = \beta_k \wedge n_k > m_k) \vee (\alpha_k > \beta_k)\,.$

$\overline{\alpha\beta} = k$

$v_\delta(\alpha) = \begin{cases} n_i & \text{if } \delta = \alpha_i\,; \\ 0 & \text{if } \delta \notin \{\alpha_1, \dots, \alpha_t\}. \end{cases}$ $\Big\|$ $\overline{\alpha\beta} = \max\{\delta : v_\delta(\alpha) \neq v_\delta(\beta)\}$

Coloring:

$$\chi^* : [\mathrm{HORD}]^3 \longrightarrow \{0, 1, 2\}$$

$$\chi^* : \{\alpha, \beta, \gamma\}_> \longmapsto \begin{cases} 0 & \text{if } \overline{\alpha\beta} > \overline{\beta\gamma} \\ 1 & \text{if } \overline{\alpha\beta} = \overline{\beta\gamma} \\ 2 & \text{if } \overline{\alpha\beta} < \overline{\beta\gamma} \end{cases}$$

$\boxed{\text{Lemma 1:}}$ Let $S = \{\alpha_1, \dots, \alpha_r\}_> \subset \mathrm{HORD}$ be χ^*-monochromatic.

1) $\chi^*(S) = 0 \;\wedge\; \alpha_1 < \omega^\omega \;\Rightarrow\; |S| = r \le N(\alpha_1) + 1$

2) $\chi^*(S) = 1 \;\Rightarrow\; |S| = r \le N(\alpha_1)$

3) $\chi^*(S) = 2 \;\Rightarrow\; |S| = r \le T(\alpha_1) + 1$

$(\alpha_1 < \gamma_{s+1} \Rightarrow |S| \le e_s(N(\alpha_1)) + 1)$

proof: 1) $\alpha_1 > \alpha_2 > \cdots > \alpha_r\,,$ $\omega^p < \alpha_1 < \omega^{p+1}, \quad N(\alpha_1) \ge p+1$

$\overline{\alpha_1\alpha_2} > \overline{\alpha_2\alpha_3} > \cdots\ \overline{\alpha_{r-1}\alpha_r}\,;$

$p \ge \overline{\alpha_1\alpha_2} > \cdots > \overline{\alpha_{r-1}\alpha_r}$

$\therefore\ r \le p+2 \le N(\alpha_1) + 1$

2)
$\overline{\alpha_1\alpha_2} = \overline{\alpha_2\alpha_3} = \cdots = \overline{\alpha_{r-1}\alpha_r}$, call it δ

$v_\delta(\alpha_1) > \cdots > v_\delta(\alpha_r) \Rightarrow v_\delta(\alpha_1) \geq r-1$

$\therefore \; N(\alpha_1) \geq r.$

3) $\overline{\alpha_1\alpha_2} < \overline{\alpha_2\alpha_3} < \cdots < \overline{\alpha_{r-1}\alpha_r} \quad T(\alpha_1) \geq r-1. \quad \square$

$\boxed{\text{Lemma 2}}$

$$\chi_2 = \chi^* : [r_2]^3 \to \{0,1,2\}.$$

If $S \subset [r_2]$ is monochromatic for χ_2, then

$$\begin{aligned} |S| &\leq \max\{N(\alpha_1)+1, T(\alpha_1)+1\} \qquad \alpha_1 = \max S \\ &\leq \max\{N(\alpha_1), e_1(N(\alpha_1))\} + 1 \\ &\leq e_1(N(\alpha_1)) + 1 = e_1(N(\max S)) + 1 \\ &= h_2(N(\max S)) \; \square \end{aligned}$$

$\boxed{\text{Lemma 3}}$ For all $s \in \mathbb{N}$, there exists a coloring

$$\chi_s : [r_s]^{s+1} \to \{0,1,\ldots,2s-2\} \text{ s.t.}$$

If $S \subset [r_s]$ is monochromatic for χ_s, then

$$|S| \leq h_s(N(\max S)).$$

proof: (By induction).

$s = 2 \Rightarrow$ Lemma 2.

$s > 2$: $\Theta = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{s+1}\}_> \subset [r_s]$ $\quad$ $(s+1)$-subset of $[r_s]$

$$\chi_s : \Theta \longmapsto \begin{cases} 2s-3 & \chi^*(\{\alpha_1,\alpha_2,\alpha_3\}) = 1 \\ 2s-2 & \chi^*(\{\alpha_1,\alpha_2,\alpha_3\}) = 2 \\ \chi_{s-1}\left[\{\overline{\alpha_1\alpha_2}, \overline{\alpha_2\alpha_3}, \ldots, \overline{\alpha_s\alpha_{s+1}}\}_>\right] & \text{when } \overline{\alpha_1\alpha_2} > \overline{\alpha_2\alpha_3} > \cdots > \overline{\alpha_s\alpha_{s+1}} \text{ all } \in r_{s-1} \\ 0 & \text{o.w.} \end{cases}$$

$S = \{\alpha_1, \dots, \alpha_r\}_> \subset [\Upsilon_S] \ =$ monochromatic.

(case 1) $\chi_s(S) = 2s-3$ or $2s-2$

$\Rightarrow \chi_{s-1}(\{\alpha_1, \dots, \alpha_{r-2}\}) = 2s-5$ or $2s-4$ etc.

$\vdots$

$\Rightarrow \chi^*(\{\alpha_1, \dots, \alpha_{r-s+2}\}) = 1$ or 2

$\Rightarrow (r-s+2) \leq \max\{N(\alpha_1), T(\alpha_1)+1\}$

$\leq \max\{N(\alpha_1), e_{s-1}(N(\alpha_1))\}+1$

$\Rightarrow \quad r \leq h_s(N(\alpha_1)) \Rightarrow |S| \leq h_s(N(\max S))$.

(case 2) $\chi_s(S) < 2s-2$

$S' = \{\overline{\alpha_1\alpha_2}, \dots, \overline{\alpha_{r-s+1}\alpha_{r-s}}\}$

$= \{\alpha_1', \dots, \alpha_{r-s+1}'\}$ $\qquad \chi^*(\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}) = 0$, if $i \leq r-s$

$\Rightarrow \alpha_1' > \dots > \alpha'_{r-s+1}$.

choose $1 \leq i_1 < \dots < i_{s-1} \leq r-s+1$ $\qquad \alpha_{i_k}' = \overline{\alpha_{i_k}\alpha_{i_{k+1}}}$

then $\chi_{s-1}(\{\alpha_{i_1}', \dots, \alpha_{i_{s-1}}'\}_>) = \chi_s(\{\alpha_{i_1}, \dots, \alpha_{i_{s-1}}, \alpha_{i_{s-1}+1}\}_>)$

$\therefore \{\alpha_1', \dots, \alpha_{r-s+1}'\} = S' = \chi_{s-1}$- monochromatic.

$r-s+1 \leq h_{s-1}(N(\overline{\alpha_1\alpha_2})) \leq h_{s-1}(N(\alpha_1))$

$\Rightarrow \quad r \leq h_s(N(\alpha_1)) \Rightarrow |S| \leq h_s(N(\max S))$.

$\square$