November 19 2013

LECTURE # 10

"There exist relatively simple problems of the theory of ordinary whole numbers that <u>cannot</u> <u>be</u> <u>decided</u> on the basis of the axioms."

Gödel, 1931.

Holds true for an extensive class of mathematical systems:

- Numbers
- Sets
- Geometry
- Topology
- Measure Theory
- Probability

First order logic underlies all of these branches of mathematics.

What Gödel showed:→

For each such system, there <u>HAD TO BE A SENTENCE</u> that asserted its unprovability in the system...

S IS <u>TRUE</u>
IFF
S IS NOT <u>PROVABLE</u>
IN THE SYSTEM.

$\Rightarrow$

$$(S = \text{TRUE} \wedge \neg \text{PROVABLE})$$
$$\vee$$
$$(S = \text{FALSE} \wedge \text{PROVABLE})$$
$$\Downarrow$$
$$(\models S \wedge \nvdash S) \vee (\not\models S \wedge \vdash S)$$
$$\Downarrow$$
$$\models \;\neq\; \vdash$$

But $\vdash \subseteq \models$ $\Rightarrow$ $\models \supsetneq \vdash$
(Soundness)

$$\Downarrow$$
$$\boxed{\models S \wedge \nvdash S}$$

$\boxed{\text{S IS TRUE AND NOT PROVABLE.}}$

SOME CAVEATS

(A) $S \equiv$ Not Liar's Paradox.

Consider $S' \equiv$ This sentence is unprovable
+ (soundness & completeness:
All proved statements are true & vice versa)
$\Rightarrow$ " S': This sentence is false".

Suppose S' is false $\Rightarrow$ S' can be proved
$\Rightarrow$ S' is true (& S' is proved to be true)
$\Rightarrow$ S' is false
$\Rightarrow$ # (Paradox)

(B) PROOF HAS TO BE WELL-DEFINED!
(System vs Meta System)

Within a given mathematical system, Σ,
the notion of a proof within that system, Σ, ~~must~~
must be well-defined.

(c) To avoid paradox, we use the following:
S: This sentence is unprovable in system Σ

S is true $\Rightarrow$ S is not provable in $\Sigma \Rightarrow \Sigma \nvdash S$ (No contradiction)
S is false $\Rightarrow$ S is provable in $\Sigma \Rightarrow \Sigma \vdash \bot \Rightarrow \Sigma$ = inconsistent.

S = true & $\Sigma \nvdash S$ $\Rightarrow$ Incompleteness (but not a paradox).

$\boxed{\text{Gödelian System}}$

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Language $\mathcal{L}$ $\left\{ \begin{array}{l} \text{Captures a wide class of} \\ \text{mathematical objects, e.g. Natural Numbers, } \mathbb{N}. \end{array} \right.$

H = Predicate in $\mathcal{L}$, e.g. $H \subseteq \mathbb{N}$, prime, perfect, etc.

$H(n)$ = sentence, defining a set.

$$\boxed{H(n) \Leftrightarrow n \in H}$$

(I) There is a well-defined set of sentences called TRUE SENTENCES.

For each sentence, S, is associated a sentence $\bar{S}$

$\equiv$ Negation of $S \equiv \neg S$

$\therefore$ For each predicate, P, is associated a predicate $\bar{P}$

$\bar{P}$: Negation of $P \equiv \neg P$

$H \equiv$ Predicate, $H(n)$ = sentence

$\overline{H(n)} \equiv \bar{H}(n) \equiv \{ n \mid n \in \mathbb{N} \setminus H \}$

(II) To each expression X a natural number n can be assigned.

$n \equiv$ Gödel number of X

Each distinct expression X has a distinct Gödel number n

Call n = sentence number iff it is the Gödel number of a sentence S

Call m = predicate number iff it is the Gödel number of a predicate P.

(III) There exists a <u>WELL-DEFINED</u> procedure for proving a sentence; the system is ~~s~~ <u>SOUND</u> iff every provable sentence is true.

A Map $\mathbb{N} \to \mathbb{N}$

n = Predicate Number

$\downarrow$

H_n = Corresponding Predicate

$\downarrow$

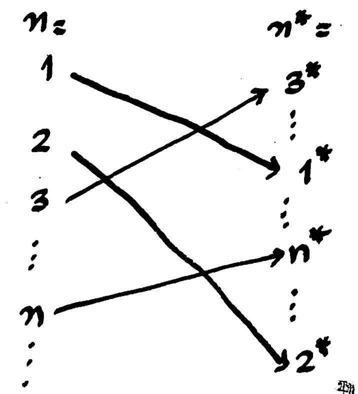
$H_n(n)$ = Sentence, which is true, iff $n \in H_n$

$\downarrow$

S_{n^*}

$\downarrow$

n^* $\boxed{n^* \in H_n \quad \text{or} \quad n^* \in \mathbb{N} \setminus H_n}$



$\boxed{\text{DIAGONALIZER}}$

A predicate $\mathcal{K}$ diagonalizes the predicate H if
$\mathcal{K}(n)$ iff $H(n^*)$ is true.

$\boxed{\text{GÖDELIAN}}$

A system Σ is said to be Gödelian iff it satisfies the following two conditions...

$\boxed{\text{G1}}$ Every predicate H has a diagonalizer $\mathcal{K}$.

$\boxed{\text{G2}}$ There is a "<u>provability</u>" <u>predicate</u> P in Σ such that for any sentence number n, the sentence $P(n)$ is true iff S_n is provable.

Start with the provabity predicate P ($\because$ G2)

$\downarrow$

Find its negation $\bar{P}$ (obeys G1)

$P(n) \Leftrightarrow \neg \bar{P}(n)$

$\downarrow$

$\bar{P}$ has a diagonalizer $\mathcal{K}$ ($\because$ G1)

$\downarrow$

$\mathcal{K}(n)$ = true iff $\bar{P}(n^*)$ = true iff $P(n^*)$ = false
iff S_{n^*} is not provable.

$\downarrow$

Let k = the Gödel number of $\mathcal{K}$, $\mathcal{K} = H_k$

$\downarrow$

$H_k(n)$ = true iff S_{n^*} = not provable $\forall n$.

$\downarrow$

$H_k(k) = \mathcal{K}(k) = \bar{P}(k^*) = \boxed{S_{k^*} = \text{true} \quad \text{iff} \quad S_{k^*} \neq \text{provable}}$

$\downarrow$

$\models_\Sigma S_{k^*}$ iff $\nvdash_\Sigma S_{k^*}$

$\downarrow$

S_{k^*} = true & not provable ($\because \vdash_\Sigma \subseteq \models_\Sigma$)

THEOREM GT (Gödel-Tarski)

For every sound Gödelian system, there must be a sentence of the system that is true, but not provable in the system.

FIXED POINT

A sentence S is called a *fixed point* of a predicate H iff
S = true **iff** $H(n)$, where n = Gödel number of S.

$H(n)$ = true iff S_n = true.
$\bar{P}(n)$ = true iff S_n = true (P = Provability Predicate)
$\downarrow$
S_n = true iff $S_n \neq$ provable. (Gödel Sentence)

THEOREM F1

In any system satisfying G1, each predicate of the system has a fixed point.

proof: H = Predicate $\rightsquigarrow$ $\mathcal{K}$ = Diagonalizer of H. ($\because$ G1)
$\mathcal{K}(k)$ = Sentence $\equiv S_{k^*}$

$H(k^*)$ = true iff $\mathcal{K}(k) = H(k^*)$ = true
iff S_{k^*} = true.
$\square$