Completeness Thm.

$\mathcal{L}$ = Language (+ signature, Σ)

c = Constant symbol.

$\mathcal{L}_c$ = The result of adjoining c to $\mathcal{L}$.

$\therefore$ $\mathcal{L}_c = \mathcal{L}$, if c already occurs in $\mathcal{L}$.

C = Set of constant symbols.

$\mathcal{L}C$: The language resulting from $\mathcal{L}$ by adjoining a set C of constants to $\mathcal{L}$.

= A CONSTANT EXPANSION OF $\mathcal{L}$.

α^c_z = Formula arising from α by replacing constant c with variable z.

$X^c_z = \{ \alpha^c_z \mid \alpha \in X \} \Rightarrow$ c no longer occurs in X^c_z.

CONSTANT ELIMINATION LEMMA (CEL)

Suppose

$$X \vdash_{\mathcal{L}_c} \alpha .$$

Then

$$X^c_z \vdash_{\mathcal{L}} \alpha^c_z$$

for almost all variables z. $\square$

$\boxed{\text{Corollary}}$ (1) Rule of Constant Quantification (RCQ)

$$\frac{X \vdash \alpha^c_x \quad (c \notin X, \alpha, \text{etc.})}{X \vdash \forall_x \alpha}$$

<u>Proof</u> of Constant Elimination Lemma (CEL).
By induction in $\vdash_{\mathcal{L}_C}$ → $\boxed{\text{Homework}}$

$\boxtimes$

$\boxed{\text{Corollary}}$ (2) Let C be any set of symbols & $\mathcal{L}' = \mathcal{L}C$.
Then, for all $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$

$$X \vdash_{\mathcal{L}} \alpha \quad \text{iff} \quad X \vdash_{\mathcal{L}'} \alpha.$$

$\square$

Thus $\vdash_{\mathcal{L}'}$ = conservative expansion of $\vdash_{\mathcal{L}}$.

<u>Abuse of Notation</u>:
$\vdash$ stands for derivability relation in $\mathcal{L}$ ... and in every constant expansion $\mathcal{L}'$ of $\mathcal{L}$.

<u>Important</u>: "X = consistent" $\equiv X \nvdash \bot$; $X \subseteq \mathcal{L}$
No need to distinguish between the consistency of X with respect to $\mathcal{L}$ or $\mathcal{L}'$.

<u>COMPLETENESS THEOREM</u>

Consistency $\Leftrightarrow$ Satisfiability
$\Updownarrow$
Model Construction.
(From the syntactic material of a certain constant expansion of $\mathcal{L}$.)

Model Construction.

For each variable x and each wff $\alpha \in \mathcal{L}$

$\Rightarrow$ Choose a constant $C_{x,\alpha}$ not occurring $\mathcal{L}$ as follows:

(★) $\alpha^x := \neg \forall x. \alpha \wedge \alpha^x_c \qquad c := C_{x,\alpha}$

Thus $\neg \alpha^x := \exists x \neg \alpha \rightarrow \neg \alpha^x_c$

Thus constant c is a counter example to the validity of α.

$\boxed{\text{Lemma}}$ Let

$$T_{\mathcal{L}} := \{ \neg \alpha^x \mid \alpha \in \mathcal{L},\ x \in var \},$$

where α^x is defined as in (★), & $X \subseteq \mathcal{L}$ is consistent.

Then $X \cup T_{\mathcal{L}}$ is consistent as well.

<u>proof</u>:

Suppose $X \cup T_{\mathcal{L}} \vdash \bot$.

Then $\exists_{n \geq 0} \exists_{\neg\alpha_0^{x_0}, \ldots, \neg\alpha_n^{x_n}} =$ wffs.

s.t.

$$X \cup \{ \neg \alpha_i^{x_i} \mid i \leq n \} \vdash \bot$$

$$X' := X \cup \{ \neg \alpha_i^{x_i} \mid i < n \} \nvdash \bot.$$

$$x := x_n, \quad \alpha := \alpha_n \quad c = C_{x,\alpha}$$

$\therefore X' \cup \{ \neg \alpha^x \} \vdash \bot$

$\Rightarrow X' \vdash \alpha^x := \neg \forall x \alpha \wedge \alpha^x_c$

$\Rightarrow X' \vdash \neg \forall x \alpha,\ \alpha^x_c$, but $X' \vdash \alpha^x_c \rightarrow \forall x \alpha$ (Ax IV)

$\Rightarrow X' \vdash \forall x \alpha$, (MP)

$\Rightarrow X' \vdash \forall x \alpha \wedge \neg \forall x \alpha \equiv \bot \Rightarrow \#$. □

HENKIN SET.

$X \subseteq \mathcal{L}$ is a Henkin Set if X satisfies the following two conditions:

(H1) $X \vdash \neg\alpha \iff X \nvdash \alpha$

(H2) $X \vdash \forall x \alpha \iff X \vdash \alpha^x_c$ for all constant c in $\mathcal{L}$.

(H1) $\wedge$ (H2) $\Rightarrow$ (H3)

For each term t there is a constant c such that

$$X \vdash t = c.$$

$\boxed{\text{Lemma H}}$ Let $X \subseteq \mathcal{L}$ be consistent (i.e. $X \nvdash \perp$). Then there exists a Henkin Set $Y \supseteq X$ in a suitable constant expansion $\mathcal{L}C$ of $\mathcal{L}$.

(L1) Every Henkin set has a model.

(L2) Every consistent set has a model.

(L3) Every consistent set is satisfiable.

$$\Longrightarrow$$

COMPLETENESS THM.

Let $\mathcal{L}$ denote any first order language. Then

$$X \vdash \alpha \iff X \models \alpha$$

for all $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$ $\square$.

Proof of Lemma H.

Let

$\mathcal{L}_0 := \mathcal{L}$, $X_0 := X$; Assume that we have already defined $\mathcal{L}_n, X_n$.

Take all $\alpha \in \mathcal{L}_n$ and $x \in Var$; Construct new constants:

$$c_{x,\alpha,n} := C_n .$$

$$\mathcal{L}_{n+1} := \mathcal{L}_n C_n .$$

$$T_{\mathcal{L}_n} := \{ \neg \alpha^x \mid \alpha \in \mathcal{L}_n ,\ x \in var \}$$

$$X_{n+1} := X_n \cup T_{\mathcal{L}_n}$$

$\therefore$ $X_{n+1} \subseteq \mathcal{L}_{n+1}$ & consistent.

Thus:

$$\forall_n \quad X_n \nvdash \bot , \ X_n \subseteq \mathcal{L}_n$$

$$X' := \bigcup_{n \in \mathbb{N}} X_n ; \ \mathcal{L}' := \bigcup_{n \in \mathbb{N}} \mathcal{L}_n ; \quad C := \bigcup_{n \in \mathbb{N}} C_n .$$

(A) $X' \nvdash \bot$ (Finiteness of proof)

(B) $\alpha \in \mathcal{L}' , x \in Var \Rightarrow \alpha \in \mathcal{L}_n$ (for minimal such n), $x \in var$

$$\Rightarrow \neg \alpha^x \in X_{n+1} \Rightarrow \neg \alpha^x \in X'$$

(C) Let $(H, \subseteq)$ be the partial order over all consistent extensions of X' in $\mathcal{L}'$

Every chain $K \subseteq H$ has an upper bound $\bigcup K$ in H. ($\because H \neq \phi$; $\because X' \in H$; Every member is consistent $\Rightarrow \bigcup K \nvdash \bot$.)

By Zorn's Lemma, H has a consistent maximal element Y.

$Y \supseteq X'$, $Y \nvdash \bot$ $\}$ Maximal

Y is a maximal consistent extension of X'

$$\neg\alpha^{x} \in X' \subseteq Y$$

$$\Rightarrow Y \vdash \neg\alpha^{x} \quad \forall_{\alpha \in \mathcal{L}'}$$

$$\Rightarrow \boxed{Y = \text{Henkin}}$$

(H1) $Y \vdash \alpha \Leftrightarrow Y \vdash \neg\alpha$ ($\because Y$ is consistent).

(H2)($\Rightarrow$)
$$\frac{Y \vdash \forall x\,\alpha, \quad Y \vdash \forall x\,\alpha \to \alpha^{x}_{c}\ (Ax\,\mathrm{II})}{Y \vdash \alpha^{x}_{c}}\ \mathrm{MP}$$
For all const. c.

($\Leftarrow$) $Y \vdash \alpha^{x}_{c} \quad \forall_{c \in \mathcal{L}'}$

$$\Rightarrow Y \vdash \alpha^{x}_{c}, \quad c := c_{x,\alpha,n} \quad \alpha \in \mathcal{L}_n$$

Suppose $Y \nvdash \forall x\,\alpha$

$$\frac{\dfrac{\Rightarrow Y \vdash \neg\forall x.\,\alpha \ (\text{by H1}); \quad Y \vdash \alpha^{x}_{c}}{Y \vdash \neg\forall x\,\alpha \wedge \alpha^{x}_{c}}}{Y \vdash \alpha^{x}}$$

but $\neg\alpha^{x} \in T_{\mathcal{L}_n}$

$$\frac{Y \vdash \neg\alpha^{x} \subseteq Y}{Y \vdash \bot \Rightarrow \#.}$$

(H3) For each term t there is a constant c s.t.

$$Y \vdash t = c$$

$Y \vdash \neg\forall_x\, t \neq x \qquad x \notin \mathrm{var}\, t$

$\Rightarrow Y \nvdash \forall_x\, t \neq x$ (by H1)

$\Rightarrow$

$Y \nvdash t \neq c$ for some c (by H2)

$\Rightarrow Y \vdash t = c$ for some c (by H1).

LEMMA Every Henkin set $Y \subseteq \mathcal{L}$ possesses a model. (TERM MODEL).

proof: Inductively construct a term model.

Let T = Term algebra of all the terms in $\mathcal{L}$.

Define an equivalence relation $\approx$ as follows:

$t \approx t'$ iff $Y \vdash t = t'$

$A := T/\approx$ (Partition of T into equivalence classes with respect to the relation $\approx$)

$\mathcal{M} := (A, \rho)$ model with ρ = v'ble assignmt.

(A) Note $\approx$ is an equivalence relation:

n-ary function f ; n-ary relation P

$t_1 \approx t_1'$; $t_2 \approx t_2'$; $\cdots$; $t_n \approx t_n'$

$\Rightarrow f\vec{t} \approx f\vec{t}'$ & $P\vec{t} = P\vec{t}'$

(B) Let C = constants in $\mathcal{L}$

Since Y = Henkin, by (H3), for each term $t \in T$, there is a c such that $c \approx t$

$\mathcal{M} := (A, \rho)$

$A := \{\bar{c} \mid c \in C\}$; $x^{\mathcal{M}} := \bar{x}$; $c^{\mathcal{M}} := \bar{c}$;

$f^{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_n) := \overline{f(t_1, \dots, t_n)}$

$P^{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_n) \Leftrightarrow Y \vdash P\vec{t}$

We need to prove

(A) $t^{\mathcal{M}} = \bar{t}$

(B) $\mathcal{M} \models \alpha \Leftrightarrow Y \vdash \alpha$.

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(A) By induction on the structure of the term $\bar{t}$

$t_i^{\mathcal{M}} = \bar{t}_i \qquad i=1,\dots,n$

$\Rightarrow \quad t = f\vec{t}$

$t^{\mathcal{M}} = f^{\mathcal{M}}(\underline{t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}}}) = f^{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_n)$

$= \overline{f(t_1, \dots, t_n)} = \bar{t}$.

(B) By induction on rk α.

$\mathcal{M} \models t=s \iff t^{\mathcal{M}} = s^{\mathcal{M}}$

$\iff \bar{t} = \bar{s}$

$\iff t \approx s \iff Y \vdash t = s$.

$\mathcal{M} \models P\vec{t} \iff P^{\mathcal{M}} t_1^{\mathcal{M}} \cdots t_n^{\mathcal{M}}$

$\iff P^{\mathcal{M}} \bar{t}_1 \cdots \bar{t}_n$

$\iff Y \vdash P\vec{t}$

$\mathcal{M} \models \neg\alpha \iff \mathcal{M} \not\models \alpha$

$\iff Y \nvdash \alpha$ (IH)

$\iff Y \vdash \neg\alpha$ (H1)

$\mathcal{M} \models \alpha \wedge \beta \iff \mathcal{M} \models \alpha$ and $\mathcal{M} \models \beta$

$\iff Y \vdash \alpha$ and $Y \vdash \beta$ (IH)

$\iff Y \vdash \alpha \wedge \beta$ (A1, A2)

$\mathcal{M} \models \forall x\, \alpha \iff \mathcal{M}_{\bar{c}}^{x} \models \alpha$ for all $c \in C$

$\iff \mathcal{M}_{c^{\mathcal{M}}}^{x} \models \alpha$

$\iff \mathcal{M} \models \alpha_c^x \quad \forall c \in C$

$\iff Y \vdash \alpha_c^x$ (IH)

$\iff Y \vdash \forall_x \alpha$ (H2) $\square$

MODEL EXISTENCE THM: Each consistent $X \subseteq \mathcal{L}$ has a model. $\square$

COMPLETENESS THM: Let $\mathcal{L}$ denote any first order language. Then $X \vdash \alpha \iff X \models \alpha$ for all $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$. $\square$