Lecture # 7.

PROOF THEORY FOR FIRST-ORDER LOGIC.

PROOFS.

{ A finite sequence of fixed indisputable steps.

PROOF ⟶ Is built from

↳ Axioms { Facts which accept without proof

↳ Rules of Inference { Theorems: Facts derived from axioms using agreed upon rules.

Thus, it is decidable whether a given sequence of steps is infact a proof.

Proof is an effective mechanism to convince a skeptic.

↳ Without enumerating all possible models and variable assignments

⟶ as could be done in propositional logic.

A DEDUCTIVE CALCULUS.

Many possible choices for Axioms & Rules of Inference.

AXIOMS. $\Lambda \leftarrow$ Possibly infinite number of axioms.

~~Rule~~ RULE OF INFERENCE Modus Ponens.

$$\frac{\alpha, \quad \alpha \to \beta}{\beta.}$$

DEDUCTION. $\Gamma \vdash \phi$

$\begin{cases} \phi \text{ is deducible from } \Gamma \\ \phi \text{ is a theorem of } \Gamma \end{cases}$

A deduction of ϕ from Γ is a sequence

$\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$ of wff's. such that

- $\alpha_n = \phi$
- For each $i \le n$ either
  - i) $\alpha_i \in \Gamma \cup \Lambda$
  - ii) for some $j, k \le i$
    α_i is obtained by modus ponens for α_j and α_k

$$\frac{\alpha_j, \alpha_k}{\alpha_i} \text{ (MP)}$$

$\begin{cases} \text{i.e. } \alpha_k \equiv \alpha_j \to \alpha_i \\ \text{or } \alpha_j \equiv \alpha_k \to \alpha_i \end{cases}$

EXAMPLE.

Axioms. (I) All tautologies.

(II) $\forall x\, \alpha \to \alpha^x_t$ $\left\{ \begin{array}{l} t \text{ is substitutable} \\ \text{for } x \text{ in } \alpha. \end{array} \right.$

(III) $\forall x\, (\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$

(IV) $\alpha \to \forall x\, \alpha$, where $x \notin$ Free in α

Prove $\quad \vdash \forall x\, (Px \longrightarrow \exists y\, Py)$

1) $\forall x \left[(\forall y\, \neg Py \to \neg Px) \to (Px \to \neg \forall y\, \neg Py) \right]$
Tautology (Contrapositivity)

2) $\forall x (\forall y\, \neg Py \to \neg Px) \to \forall x (Px \to \neg\forall y\, \neg Py)$
MP 1; III

3) $\forall x\, (\forall y\, \neg Py \to \neg Px)$ II & IV & MP.

4) $\forall x\, (Px \to \neg\forall y\, \neg Py)$ MP (2; 3)

$\equiv \forall x\, (Px \to \exists y\, Py)$

A set of formulas Δ is <u>closed under modus ponens</u> iff

$$\alpha \in \Delta ; \ \alpha \to \beta \in \Delta$$

implies that $\beta \in \Delta$.

INDUCTION PRINCIPLE.

S = Set of wff's.

$\Gamma \cup \Lambda \subseteq S$ and S is closed under MP

Then S contains every theorem of Γ.

$\boxed{\text{AXIOMS.}}$

A wff ϕ is a <u>generalization</u> of ψ iff for some variables $x_1, \ldots, x_n$, where $n \geq 0$, we have

$$\phi = \forall x_1 \cdots \forall x_n \, \psi.$$

The axioms Λ are ~~made~~ made up of all generalizations of wff's of the following forms, where x & y are variables and α & β are wff's.

AXIOMS (Contd.)

I) TAUTOLOGIES.

II) $\forall x \; \alpha \to \alpha^x_t$, where t is substitutible for x in α.

III) $\forall x \; (\alpha \to \beta) \to (\forall x \, \alpha \to \forall x \, \beta)$

IV) $\alpha \to \forall x \; \alpha$

V) $x = x$

VI) $x = y \to (\alpha \to \alpha')$, where α = atomic and α' is obtained from α by replacing x in zero or more places by y.

Note V & VI assumes that the language includes equality.

I) TAUTOLOGIES: Wff's obtained from tautologies of propositional logic by replacing each propositional symbol by a wff of First-order logic.

$$(A \to \neg B) \to (B \to \neg A)$$

$$(\forall y \, Qy \to \neg Px) \to (Px \to \neg \forall y \, Qy)$$

$$\forall x (\forall y \, Qy \to \neg Px) \to (Px \to \exists y \neg Qy)$$

A first-order formula is prime if it is atomic or of the form $\forall x \, \alpha$.

Soundness Theorem.

If $\Gamma \vdash \phi$ then $\Gamma \models \phi$.

Proof in 3 steps.

Substitution Lemma:

If the term t is substituted for the variable x in the wff ϕ, then for ~~every~~ any model $\mathcal{M}$ and variable assignment ρ,

$$\mathcal{M} \models_{\rho} \phi^{x}_{t} \quad \text{iff} \quad \mathcal{M} \models_{\rho(x \mid \bar{\rho}(t))} \phi.$$

{ ⇒ If we replace a variable x with a term t, the semantics are same as if the variable assignment is modified so that x takes on the same value as the term t. }

Induction on the structure of the wff ϕ.

GENERALIZATION THEOREM.

If $\Gamma \vdash \phi$ and x does not occur free in any formula in Γ then

$$\Gamma \vdash \forall x \phi.$$

Proof: Show that

$$\Gamma \cup \Lambda \subseteq \{\phi \mid \Gamma \vdash \forall x \phi\} = S \text{ is}$$

closed under modus ponens.

Case 1. Suppose $\phi \in \Gamma$. Then x does not occur free in ϕ. By Axiom Gp IV:

$$\phi \to \forall x \phi$$

$$\Rightarrow \quad \frac{\Gamma \vdash \phi \qquad \Gamma \vdash \phi \to \forall x \phi}{\Gamma \vdash \forall x \phi} \text{ MP.}$$

Case 2. Suppose $\phi \in \Lambda$ (tautology)

$\Rightarrow \forall x \phi$ is a tautology & is an axiom.

$$\therefore \Gamma \vdash \forall x \phi.$$

Case 3. Suppose ϕ is derived by M.P.

$$\frac{\Gamma \vdash \psi \; ; \; \Gamma \vdash \psi \to \phi}{\Gamma \vdash \phi}$$

By induction $\Gamma \vdash \forall x \psi$

$\Gamma \vdash \forall x (\psi \to \phi)$

But $\Gamma \vdash \forall x(\psi \to \phi) \to \forall x \psi \to \forall x \phi$ (Axiom Gp III)

$$\frac{\Gamma \vdash \forall x \psi \; ; \; \Gamma \vdash \forall x \psi \to \forall x \phi}{\Gamma \vdash \forall x \phi.}$$

DEDUCTION THEOREM.

If $\Gamma \cup \{\gamma\} \vdash \phi$ then $\Gamma \vdash (\gamma \to \phi)$

Proof. $\Gamma \cup \{\gamma\} \vdash \phi$

iff $\Gamma \cup \{\gamma\} \cup \Lambda$ tautologically implies ϕ

iff $\Gamma \cup \Lambda$ tautologically implies $\gamma \to \phi$

iff $\Gamma \vdash \gamma \to \phi$.

Note A tautologically implies B
$\Rightarrow$ Whenever A is true B must be true.

SOUNDNESS THEOREM

$$\Gamma \vdash \phi \Rightarrow \Gamma \models \phi.$$

Proof
(By Induction).

Case 1. ϕ is a logical axiom
$\therefore \models \phi$ (Check: All axioms are SOUND).
Thus. $\Gamma \models \phi$.

Case 2. $\phi \in \Gamma$. Thus $\Gamma \models \phi$ (by defn).

Case 3.

$$\frac{\Gamma \vdash \psi; \ \Gamma \vdash \psi \to \phi}{\Gamma \vdash \phi} \text{ M.P.}$$

By Ind. Hyp.

$\Gamma \models \psi$ and $\Gamma \models \psi \to \phi \equiv \Gamma \models \neg\psi \vee \phi$

Thus. $\Gamma \models \psi \wedge (\neg\psi \vee \phi)$

$\models (\psi \wedge \neg\psi) \vee (\psi \wedge \phi)$

$\models \psi \wedge \phi$

$\Gamma \models \phi$.

Checking that all axioms are sound.
All trivial, except <u>Axiom Gp II</u>.

$$\forall x\, \alpha \to \alpha^{x}_{t}$$ where t is substituted for x in α.

Suppose $\mathcal{M} \models_{\rho} \forall x\, \alpha$

Thus $\forall d \in \mathrm{dom}(\mathcal{M})$ $\mathcal{M} \models_{\rho(x|d)} \alpha$

Make $d = \bar{\rho}(t)$

Then $\mathcal{M} \models_{\rho(x|\bar{\rho}(t))} \alpha$

By substitution lemma

$\mathcal{M} \models \alpha^{x}_{t}$. □

$\boxed{\text{COMPLETENESS THEOREM}}$ (Gödel 1930).

If $T \models \phi$ then $T \vdash \phi$.

Show that any consistent set of formulas is satisfiable.