Lecture #5

Two notions. $\models$ and $\vdash$

What's the relation between them?

- Satisfiability $\models$ (Model Checking) $\Big\}$ $X \models \alpha$.
- Provability $\vdash$ (Theorem Proving) $\Big\}$ $X \vdash \alpha$

<u>Soundness</u>

$X \vdash \alpha$ implies $X \models \alpha$.

(Proof by induction on formula and rule)

<u>Completeness</u>

$X \nvdash \alpha$ implies $X \not\models \alpha$

Strengthen the statement:

$X \nvdash \alpha \wedge \underset{(X = \text{consistent})}{X \nvdash \bot}$ implies $X \not\models \alpha$

$$\left.\begin{array}{c} X \vdash \bot \\ \hline X \vdash p \wedge \neg p \\ \hline X \vdash p \mid X \vdash \neg p \\ \hline X \vdash \alpha \quad \forall \alpha \end{array}\right\}$$

Note $X, \neg\alpha \vdash \bot$

$$\left.\dfrac{\dfrac{X, \neg\alpha \vdash \bot}{X, \neg\alpha \vdash \alpha} \qquad \dfrac{\alpha \vdash \alpha}{X, \alpha \vdash \alpha}}{X \vdash \alpha}\right\}$$

Hence

$X \cup \{\neg\alpha\}$ = consistent <u>implies</u> $X \not\models \alpha$

$Y \supseteq X \cup \{\neg\alpha\}$ = Maximally consistent superset of X

$\Rightarrow$ Y = satisfiable

$\Rightarrow$ X = satisfiable.

Completeness Proof:

$X \nvdash \alpha$ ($\wedge$ X = consistent)

$\Rightarrow$ $X \cup \{\neg\alpha\}$ = consistent

$\boxed{\text{Lindenbaum's Lemma}}$

$\Rightarrow$ $\exists Y$ $Y \supseteq X \cup \{\cancel{\alpha}\neg\alpha\}$ & Y = maximally consistent

$\Rightarrow$ Y = satisfiable

$\Rightarrow$ $X \cup \{\neg\alpha\}$ = Satisfiable

$\Rightarrow$ $X \not\models \alpha$

Gentzen Rules.

(IS) $\dfrac{}{\alpha \vdash \alpha}$ (MR) $\dfrac{X \vdash \alpha}{X' \vdash \alpha}$ $(X' \supseteq X)$

($\wedge$1) $\dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$ ($\wedge$2) $\dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$

($\neg$1) $\dfrac{X \vdash \alpha, \neg\alpha}{X \vdash \beta}$ ($\neg$2) $\dfrac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta}$

Defn

$X \subseteq \mathcal{F}$ is called inconsistent if $X \vdash \alpha$ $\forall \alpha \in \mathcal{F}$; otherwise, consistent.

$Y \subseteq \mathcal{F}$ is called maximally consistent if Y is consistent but each $Z \supsetneq Y$ is inconsistent.

Lindenbaum's Lemma

Every consistent set $X \subseteq \mathcal{F}$ can be extended to a maximally consistent set $X' \supseteq X$.

Proof.

Let H be the set of all consistent $Y \supseteq X$, partially ordered w.r.t. $\subseteq$.

$$H = \{ Y \mid Y \supseteq X \ \& \ Y \nvdash \bot \}$$

(a) $H \neq \emptyset$ $(\because X \in H)$

(b) $\exists K \subseteq H$ K = chain

I.e. $\forall Y, Z \in K$ $Y \subseteq Z$ or $Z \subseteq Y$

$\mathcal{U} := \bigcup K$ = upperbound for K.

$\mathcal{U} \nvdash \bot$

suppose not $\mathcal{U} \vdash \bot$

$\Rightarrow \mathcal{U}_0 \vdash \bot$ $\mathcal{U}_0$: finite $= \{\alpha_0, \cdots, \alpha_n\}$

$\alpha_0, \cdots, \alpha_n \vdash \bot$

$\alpha_i \in Y_i \in K$

& Y is the biggest among $Y_0, \cdots, Y_n$

$\Rightarrow \{\alpha_0, \cdots, \alpha_n\} \subseteq Y$ & $Y \vdash \bot$ (MR)

$$\frac{\alpha_0, \cdots, \alpha_n \vdash \bot}{Y \vdash \bot}$$

$\Rightarrow Y \notin H \Rightarrow \#$

Zorn's Lemma:

H has a maximal element

X' = maximally consistent. $\square$

Y = Maximally consistent
$\Rightarrow$ Y = Satisfiable.

Define ω such that

$$\omega \models p \quad \text{iff} \quad Y \vdash p \quad \forall p = \text{prime variable}.$$

Show that $\forall_\alpha \; Y \vdash \alpha$ iff $\omega \models \alpha$
(i.e. ω = model for Y).
$\Rightarrow$ Y = satisfiable.

$Y \vdash \alpha \wedge \beta$ iff $Y \vdash \alpha, \beta$ ($\wedge 1$ & $\wedge 2$)
iff $\omega \models \alpha, \beta$
iff $\Leftrightarrow$ $\omega \models \alpha \wedge \beta$

$Y \vdash \neg\alpha$ iff $Y \nvdash \alpha$ ($\because$ maximality of Y.
iff $\omega \not\models \alpha$
iff $\omega \models \neg\alpha$ $\square$

($\Rightarrow$) $\dfrac{Y \vdash \neg\alpha \wedge Y \vdash \alpha}{Y \vdash \bot}$

($\Leftarrow$) $Y \nvdash \alpha$
$\Rightarrow Y \cup \{\neg\alpha\}$ is a consistent extension of Y
$\Rightarrow \neg\alpha \in Y \cup \{\neg\alpha\}$
$\Rightarrow Y \vdash \neg\alpha$

$\neg\alpha \in Y$
(Y's maximal!)

FIRST ORDER LOGIC.

More powerful than first order logic.
(Expressive / Computationally complex)

Syntax

Expressions in first-order logic are made up of a sequence of symbols.

1) Logical Symbols
2) Parameters (Nonlogical Symbols)

Logical Symbols :

- Parantheses : (,)
- Propositional connectives : $\neg$, $\wedge$
- Variables : v_1, v_2, ...
- Quantifiers : $\forall$ (universal quantifier, For All)

$\rightarrow \exists \equiv \neg \forall \neg$: Existential quantifies, There Exists

Parameters

- Equality : =
- Predicate Symbols : $p(x)$, $x > y$
- Function Symbols : $f(x)$, $x + y$
- Constant Symbols : 0, π, Kurt

Arity: Predicates and Functions have an arity.
→ A natural number indicating how many arguments it takes.

=; has arity = 2
2-ary predicate
constant; has arity = 0
0-ary function

A first-order language must specify its parameters.

Changing the parameters changes the language.

Examples.

	Prop. Logic	Set Theory	Elementary Number Theory
Equality	No	yes	Yes
Predicate	$P_1, P_2, \cdots$ (0-ary)	$\in$	$<$
Function	None	None	$S, +, \times, \exp$
Constant	None	ϕ	0

TERMS

For each function symbol f of arity n, we define a term-building operation $\mathcal{F}_f$

$$\mathcal{F}_f(\alpha_1, \alpha_2, \dots, \alpha_n) = f\alpha_1\alpha_2\cdots\alpha_n$$

The set of terms is the set of expressions generated from the constant symbols and variables by the $\mathcal{F}_f$ operations.

$$\langle\alpha\rangle := const \mid v_1 \mid v_2 \mid \cdots \mid \mathcal{F}_f(\langle\alpha_1\rangle, \langle\alpha_2\rangle, \cdots \langle\alpha_n\rangle)$$

FORMULAS

Atomic Formulas.

An atomic formula is an expression of the form

$$P t_1 t_2 \dots t_n$$

where P is a predicate symbol of arity n and $t_1, t_2, \dots, t_n$ are terms.

Formulas

Formula building operations:

$$\mathcal{E}_\neg(\alpha) := (\neg\alpha)$$

$$\mathcal{E}_\wedge(\alpha, \beta) := (\alpha \wedge \beta)$$

$$\mathcal{Q}_i(\alpha) := \forall_{v_i}\, \alpha$$

WFF

The set of <u>well-formed formulas</u> (<u>wffs</u> or just <u>formulas</u>) is the set of expressions generated from atomic formulas by the operations

$$\mathcal{E}_\neg \;,\; \mathcal{E}_\wedge \;,\; \mathcal{Q}_i \quad (i = 1, 2, \dots n)$$

Examples.

$$\forall_{v_1} \; \exists_{v_2, v_3} \; (v_3 > 0) \wedge (\cancel{v_2 + v_2} \; v_1 = v_2 + v_3)$$

Actually

$$\forall_{v} \; \exists_{v_2 v_3} \; (> v_3 \, 0) \wedge (= v_1 + v_2 v_3)$$

Free and Bound Variables

A variable x occurs <u>free</u> in a wff α

- If α is an atomic formula, then x occurs free in α <u>iff</u> x occurs in α
- x occurs free in $(\neg\alpha)$ iff x occurs free in α.
- x occurs free in $(\alpha \wedge \beta)$ iff x occurs free in α or in β.
- x occurs free in $\forall_{v_i} \alpha$ iff x occurs free in α and $x \neq v_i$.

BOUND

If $\forall v_i$ appears in α, then v_i is said to be bound in α.

Note: A variable can occur both free and bound in a wff α.

$$\alpha \equiv \cancel{\forall} (v_1 = v_2) \wedge \forall_{v_2} \exists v_3 (v_1 = v_2 + v_3)$$

v_1 = free

v_2 = free & bound

v_3 = bound

Rename.

$$\alpha' = (v_1 = v_2) \wedge \forall_{v_4} \exists_{v_3} (v_1 = v_4 + v_3)$$

v_1, v_2 = free v_3, v_4 = bound.

α' CLEARER THAN α.

We will require that the sets of free and bound variables in a wff are disjoint.

SENTENCE

If no variable occurs free in a wff α, then α is a sentence.

Substitutions

* $(\ )^x_t$ of some term t for a single variable x

Simple Substitution

$$\varphi^{x}_{t} \equiv \varphi_x(t) \equiv \text{'}\varphi\ t \text{ for } x\text{'} \qquad \text{occasionally } \left[\varphi^{\frac{t}{x}}\right]$$

The formula that results from replacing all free occurrences of x in φ by the term t.

$$x^{x}_{t} = t$$
$$y^{x}_{t} = y \qquad (x \neq y)$$
$$c^{x}_{t} = c \qquad c = \text{constant}$$
$$(f\ \alpha_1 \cdots \alpha_n)^{x}_{t} \equiv f\ \alpha_1{}^{x}_{t} \cdots \alpha_n{}^{x}_{t}$$
$$(P\ t_1 \ldots t_n)^{x}_{t} \equiv P\ t_1{}^{x}_{t} \ldots t_n{}^{x}_{t}$$
$$(t_1 = t_2)^{x}_{t} \equiv t_1{}^{x}_{t} = t_2{}^{x}_{t}$$
$$(\neg\alpha)^{x}_{t} \equiv \neg(\alpha^{x}_{t})$$
$$(\alpha \wedge \beta)^{x}_{t} \equiv \alpha^{x}_{t} \wedge \beta^{x}_{t}$$
$$(\forall v_i\, \alpha)^{x}_{t} \equiv \begin{cases} \forall v_i\ \alpha & x = v_i = \text{Bound v'ble} \\ \forall v_i\ \alpha^{x}_{t} & x = \text{free} \end{cases}$$

SIMULTANEOUS SUBSTITUTIONS

$\varphi^{x_1\ x_2\ \cdots\ x_n}_{t_1\ t_2\ \cdots\ t_n}$ $\quad (x_1, x_2, \cdots, x_n$ distinct$)$

The variables x_i are simultaneously replaced by the term t_i at free occurrences.

<u>Example</u> <u>First Order Theory of Numbers</u>.

Model $(\mathcal{N}, 0, S, +, \times)$

- The ordering relation
  $\{\langle m,n\rangle \mid m < n\}$
  is defined by
  $v_1 < v_2$ iff $\exists v_3\ (v_1 + S v_3 = v_2)$
- For any natural number n, $\{n\}$ is definable.
  $0, S0, SS0, SSS0, \cdots$
  $0 \in \mathcal{N}$ $\quad v_1 \in \mathcal{N}$ iff $v_1 = 0$ or
  $\exists v_2 \in \mathcal{N}\ v_1 = S v_2$
- The set of prime is definable, $\mathcal{P} \subseteq \mathcal{N}$
  $v_1 \in \mathcal{P}$ iff $v_1 > 1$
  $\wedge\ \forall v_2 \forall v_3\ (v_1 = v_2 \times v_3$
  $\rightarrow v_2 = 1 \vee v_3 = 1)$
- Note some relations on $\mathcal{N}$ are not definable.