Logic

1) History of Logic

2) Notational Things.

Office Hrs Mon 2pm.

J.R. Shoenfield, Mathematical Logic. 2001.
H. Enderton, A Mathematical Introduction to Logic, 2001.

Syllabus.

1) Propositional Logic.
2) First Order Logic.
3) Incompleteness and Undecidability
4) Second Order Logic, Many-Sorted Logic & Modal Logic.

Lecture #1.

Rough Summary. (1)

Based on Lecture by Moshe Vardi

Philosophical Logic

Aristotle. (stoic)
Epimenidis (of Crete) } Greek: Philosophy vs. Sophistry.

Liar's Paradox:

"All cretans are liars." "I am a liar."

"This statement is false."

Constructivists.

Ramon Lull (1290) – Lullian Circle

Lingua characteristica Universalis

Leibnitz (1646-1716) - Universal Language

George Boole (1815-1864) - Logic Algebraic

$x = xx$ idempotent

$x(1-x) = 0$ LEM

Jevons (1835-1882) - Logic Machine

Logical Piano

Claude Shanon (1916-2001) Relay cts.

Peirce/Markan (1889) – Logic → Calculation.

A Logical Machine

Words ought to be a little wild,
for they are assault of thoughts upon the unthinking.
- Keynes.

Mathematical Logic [Deductive Logic]

Georg Cantor (1874) — Self-reference.

Peano — Frege (1879) — ~~Be~~ Begriffschrift [conceptscript]

Rusell (1872-1970) — Principia Mathematica. Russell + Whitchead.

David Hilbert (1862-1943) — 27 open problems. Grundlagen der Mathematik — Hilber/Bernays

Gödel — 1^{st} & 2^{nd} Inc. Thm.

Turing / Church. — Consistency of 1^{st} order formula.

von Neumann (1903-1957).

Wir müssen wissen
Wir werden wissen

We must know
We will know

$$\left.\begin{aligned} t &= \{ s \mid s \notin s \} \\ t \in t &\Rightarrow t \notin t \\ t \notin t &\Rightarrow t \in t \end{aligned}\right\} \text{Russell's Paradox.}$$

Applications

- Mathematics
- Computer Science
- Linguistics
- Game Theory
- Philosophy

Definition.

Notation.

Modern Logic.

Object Language .... Meta Language
(Formalizable) (Semiformalizable)

Proof Theory ..... Model Theory
(Syntax) (Semantics)

(Algorithms / complexity) $\longrightarrow$ Applications

SAT Solver
Theorem Prover
} { Model Checker.

Set Operations: $\cup$ (union)
$\cap$ (intersection)
$\setminus$ (complement)

$\mathbb{N}$ = Set of natural numbers. (including 0)
$\mathbb{Z}$ = Set of integers.
$\mathbb{Q}$ = Set of rational numbers
$\mathbb{R}$ = Set of real numbers.

$\mathbb{N}_+, \mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ { Set of positive numbers of the corresponding sets.

n, m, i, j, k = Variables ranging over $\mathbb{N}$.

M, N = Sets.

$M \subseteq N$ inclusion

$M \subset N$ proper inclusion

$M \setminus N$ set difference

(If M = fixed, write $\setminus N$ or $\neg N$)

ϕ = Empty set.

$\mathcal{P}M = \{ s \mid s \subseteq M \}$ = Power set of M.
Set of all ~~sets~~ subsets.

Relation

A relation between M and N is a subset of $M \times N$

$\equiv$ The set of ~~all~~ order pairs (a, b)
$a \in M$ and $b \in N$.

Function

A function or mapping from M to N is a relation $f \subseteq M \times N$, if for each $a \in M$ there is possibly one $b \in N$ with $(a, b) \in f$.

$\cancel{\forall_{a \in M} \{ (a,b) \in f}$

$$\forall_{a \in M} \left| \{ b \mid (a, b) \in f \} \right| \leq 1 .$$

$b \equiv f(a)$ = Value of f at a

$f : M \to N$

$: a \mapsto f(a)$

$f : M \to N$

$: x \mapsto t(x)$ provided $f(x) \equiv t(x)$ for some term t.

$\mathrm{dom}\, f \equiv M =$ Domain of f.

$\mathrm{ran}\, f \equiv \{ f(x) : x \in M \} \subseteq N$

$=$ Range of f.

$id_M : M \to M$ identity function on M.

$: x \mapsto x$

$f : M \to N$

Injective (ONE-TO-ONE) if $f(x) = f(y) \Rightarrow x = y \;\forall x, y \in M$

Surjective (ONTO) if $\mathrm{ran}\, f = N$ $\boxed{\forall y \in N \;\exists x \;\; f(x) = y}$

Bijective (BOTH ONE-TO-ONE & ONTO) if f is both injective and surjective.

$M^I \equiv$ The set of all functions from the set I to M

$\equiv \{ f : I \to M \}$

Let f and g be two functions s.t.

$$\operatorname{ran} g \subseteq \operatorname{dom} f$$

$$h : \operatorname{dom} g \to \operatorname{ran} f$$
$$: x \mapsto f(g(x))$$

is called their <u>composition</u> (product).

$$h \equiv f \circ g$$

If A is an <u>alphabet</u> (i.e., if the elements $s \in A$ are <u>symbols</u> or <u>named symbols</u>) then the sequence

$$(s_1, s_2, \dots, s_n) \in A^n$$

$\left\{ \begin{array}{l} \text{map} \\ f : \{1, n\} \to A \end{array} \right.$
index set to its range.

is written as

$$s_1 s_2 \dots s_n$$

and is called a <u>string</u> or <u>word</u> over the <u>alphabet</u> A.

Empty string = ϕ (empty sequence)
Atomic string (a single symbol)

Let $\xi\eta$ denote concatenation of the strings ξ and η.

Let $\xi = \xi_1 \eta \xi_2$ for some strings ξ_1, η and ξ_2.

$\eta \neq \phi \Rightarrow$ η is called a *substring* (or segment) of ξ.

{ η = is called a proper substring if $\xi \neq \eta$.

$\xi_1 = \phi \Rightarrow$ η is called a *prefix* (or initial) of ξ.

$\xi_2 = \phi \Rightarrow$ η is called a *suffix* (or final) of ξ.

PREDICATE

Subsets $P, Q, R, \dots \subseteq M^n \equiv \underbrace{M \times M \times \cdots \times M}_{n}$

are called <u>n-ary predicates</u> of M.
(n-ary relation)

A <u>unary predicate</u> $\equiv$ Subset of M.

$P\vec{a}$ = True if $\vec{a} \in P$
$\neg P\vec{a}$ = True if $\vec{a} \notin P$.
(or $P\vec{a}$ = False)

An <u>n-ary operation</u> of M is a function

$$f : M^n \to M$$

Note: $M^0 = \{\phi\}$

A 0-ary operation of M is of the form

$$\{(\phi, c)\} \text{ with } c \in M$$

$\equiv$ Denoted by c and is called a <u>constant</u>.

graph $f = \{(a_1, a_2, \cdots, a_n, a_{n+1}) \mid f(a_1, a_2, \cdots, a_n) = a_{n+1}\}$ = Predicate $\subseteq M^{n+1}$

An operation $f: M^n \to M$ is uniquely described by the graph f.

Binary operation on a set A

$\circ: A^2 \to A.$

Commutative if $a \circ b = b \circ a \quad \forall a, b \in A.$

Associative if $a \circ (b \circ c) = (a \circ b) \circ c \quad \forall a, b, c \in A.$

Idempotent if $a \circ a = a \quad \forall a \in A$

Invertible if $\forall a, b \in A \ \exists x, y \in A$

$a \circ x = b \wedge y \circ a = b.$

Metalanguage.

Expressions in our metalanguage:

H, Θ

$H \Leftrightarrow \Theta$	H iff Θ	Boolean Conjuctives.
$H \Rightarrow \Theta$	if H then Θ	
$H \wedge \Theta$	H and Θ	
$H \vee \Theta$	H or Θ	

Propositional Logic.

Two-Valued PL: $\begin{cases} \text{True, False} \\ \mathrm{T}, \mathrm{F} \end{cases}$

$\begin{cases} \text{Top, Bottom} \\ \top, \bot \end{cases}$

$\begin{cases} 0, \ 1 \\ \text{zero one.} \end{cases}$

$A, B \ldots$ = Sentences in PL.

Propositional Logic studies analysis of connections of given sentences...

$A \wedge B$	A and B	not A	$\neg A$
$A \vee B$	A or B	if A then B	$A \Rightarrow B$

Modes ... Modal Logic $\begin{cases} \text{Local features} \\ \text{Temporal features} \end{cases}$

$\Diamond A$, $\Box A$, $A \cup B$

A until B
Here A there B
Necessarily A Possibly A
Sometimes A Alway A

Many-valued Logic.

Non-classical Logic.

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FORMAL LOGIC.

Two fundamental principles:

1) Principle of Bivalence { Only two truth values exist. namely: true/false

LEM: Law of Exclusive Middle.

2) Principle of Extensionality

{ The truth value of a connected sentence depends only on truth values of its parts. (not their meaning).

There is at least one snark
There is at most one snark
Every snark is a boojum

$\exists x \quad S(x)$
$\forall x \forall y \quad S(x) \wedge S(y) \Rightarrow x = y$
$\forall x \quad S(x) \Rightarrow B(x)$

$\cancel{\exists x \; B(x)}$ There must be a Boojum
$\exists x\, B(x)$.

Degrees of Truth }
Sense-Content } $\boxed{\text{Ignored}}$.