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Idea: Once we have contracted some number of
edges, the probability that we contract an edge
in the minimum cut is pretty large,
so we should be more careful the longer we
go on.

Suppose we have contracted \( m-t \) edges.

\[ \Rightarrow \text{there are } t \text{ edges left.} \]

What is the probability that we have the cut
still present in the graph?

\[ \Rightarrow \text{At each step we should not have contracted} \]

any edge in the cut

\[
= \prod_{i=1}^{m-t} \left( 1 - \frac{2}{n-i-1} \right) = \frac{t(t-1)}{n(n-1)}
\]
\[ t = \frac{u}{\sqrt{a}} \]

\[ \frac{t(t+1)}{n(n-1)} \approx \frac{1}{2} \]

So keep contracting till we reach \( \lceil \frac{u}{\sqrt{2}} \rceil + 1 \) vertices & then split into 2 processes to maximize probability.

More specifically \( \Rightarrow \) generate 2 graphs of size \( \lceil \frac{u}{\sqrt{2}} \rceil + 1 \) by contracting random edges.

& find the solution for these 2.

Contract \((d_g, t)\)

while \(|V| > t\)

\[ e \leftarrow \text{random edge} \]

\[ d_g \leftarrow d_g / e \]

return \(d_g\)

Rec-Contract \((d_g)\)

if \(V \leq 6\)

recompute force \((d_g)\)

else

\[ d_{g1} = \text{Contract} \left( d_g, \lceil \frac{u}{\sqrt{2}} \rceil + 1 \right) \]
\[ g_2 = \text{Contract}(g_1, \left\lceil \frac{u}{\sqrt{2}} \right\rceil + 1) \]
\[ \text{return } \min(\text{Rec-Contract}(g_1), \text{Rec-Contract}(g_2)) \]

Claim: \text{Rec-Contract} finds a min-cut with probability \( \Theta\left(\frac{1}{\log n}\right) \)

Proof:

the cut needs to survive all the way down to the 2 nodes, but it only needs to survive in one of the calls. Let \( P(u) = \text{prob that we get it} \).

Base case if \( u \leq 6 \)

then \( P(u) \geq \frac{1}{15} \) \( \rightarrow \) \( \frac{1}{2} P\left(\left\lceil \frac{u}{\sqrt{2}} \right\rceil + 1\right) \)

else \( P(u) \geq 1 - \left(1 - \frac{1}{2} P\left(\left\lceil \frac{u}{\sqrt{2}} \right\rceil + 1\right)\right)^2 \)

\( u \rightarrow \left\lceil \frac{u}{\sqrt{2}} \right\rceil + 1 \rightarrow \ldots \rightarrow 1 \)

\( P_0 \rightarrow P_1 \rightarrow \ldots \rightarrow P_{k-1} \rightarrow P_k \rightarrow P_0 \)

\( P_k \rightarrow \text{probability at k\textsuperscript{th} level of recursion} \)

\[ P_{k+1} \geq 1 - (1 - \frac{1}{2} P_k)^2 \]
\[ \Rightarrow P_{k+1} \geq P_k - \frac{1}{4} P_k^2 \]

We will prove that \( P_k \geq \frac{1}{k+1} \) by induction.

**Base case** \( k = 0 \): \( P_0 = \frac{1}{1} \), obvious as if we are on leaf.

We get the min-cut of current graph with Prob = 1.

**I.H.** \( P_k \geq \frac{1}{k+1} \)

\[ P_{k+1} \geq P_k - \frac{1}{4} P_k \]

\[ = \frac{1}{k+1} - \frac{1}{4(k+1)^2} \]

\[ = \frac{4k+3}{4(k+1)^2} \geq \frac{1}{k+2} \]

As \((4k+3)(k+2) = 4k^2 + 11k + 6 > 4k^2 + 8k + 4 = 4(k+1)^2\)

\[ \Rightarrow \text{I.H is true for } \forall k \]

\[ \Rightarrow P_n \geq \frac{1}{n+1} = \frac{1}{\log_2 n + 1} \]

\[ \Rightarrow \text{Rec - contract succeeds with probability at least } \frac{1}{\log_2 n + 1} \approx \frac{1}{\log_2 n} \]
\[
T(n) = 2 T \left( \frac{n}{\sqrt{2}} \right) + O(n^2)
\]

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\]

\(L\) adjacency matrix.

\(L\) adjacency list

\[
T(n) = O(n^2 \log n) \text{ by master thm.}
\]

if we repeat this \(O(\log n)\) times we get

success probability \(= \left(1 - \frac{1}{\log n} \right)^{\log n} \approx \frac{1}{e}
\]

In general for graph algorithms.

we need success probability = \(O\left( \frac{1}{\text{poly}(n)} \right) \)

so to get \(\frac{1}{n}\) probability of success

\[
\frac{1}{n} = \left( \frac{1}{e} \right)^{\log n}
\]

\[
\Rightarrow \text{ repeat} \, \log^2 n \text{ times}
\]

\[
\left[ \left(1 - \frac{1}{\log n} \right)^{\log n} \right]^\log n = \left( \frac{1}{e} \right)^{\log n} = \frac{1}{n}
\]
Chebyshev's Inequality

In general, only knowing the mean of a distribution is not enough. Also important is how the distribution is spread.

**Variance / Standard Deviation**

\[ \mathbb{E} \left[ (X - \mathbb{E}[X])^2 \right] = \sigma^2 \quad \Rightarrow \quad \text{variance} \]

\[ \mathbb{E} \left[ X^2 - 2X \mathbb{E}[X] + \mathbb{E}[X]^2 \right] \]

\[ = \mathbb{E}[X^2] - 2 \mathbb{E}[X]^2 + \mathbb{E}[X]^2 \]

\[ = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \]

**Example:**

\( X \rightarrow \text{uniform over } \{-1, 1\} \)

\( \mathbb{E}[X] = 0 \)

\( \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \)

\[ = \mathbb{E}[X^2] = 1 \]
Variance of binomial

\[ \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \]

\[ = \mathbb{E}[X^2] - (np)^2 \]

\[ \mathbb{E}[X^2] = \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \]

Linearity of variance for independent variables

\[ \text{Var}(X+Y) = \mathbb{E}[(X+Y - \mathbb{E}[X+Y])^2] \]

\[ = \mathbb{E}[X^2 + Y^2 + \mathbb{E}[X]^2 + \mathbb{E}[Y]^2 - 2\mathbb{E}[X] \cdot Y + 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + 2\mathbb{E}[Y] \cdot X] - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + 2\mathbb{E}[Y] \cdot \mathbb{E}[X] - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - 2\mathbb{E}[Y] \cdot \mathbb{E}[X] \]

\[ = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + \mathbb{E}[X]^2 + \mathbb{E}[Y]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + 2\mathbb{E}[Y] \cdot \mathbb{E}[X] - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - 2\mathbb{E}[Y] \cdot \mathbb{E}[X] \]

\[ = \mathbb{E}[X^2] - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] + \mathbb{E}[Y]^2 - 2\mathbb{E}[Y]^2 - 2\mathbb{E}[X]^2 \]

\[ = \mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y]^2 - \mathbb{E}[Y]^2 \]

\[ = \text{Var}(X) + \text{Var}(Y) \]
\[
\text{Var (Binom) = } n \cdot \text{Var (Bernoulli)}
\]

\[
\text{Var (Bernoulli)} = E[B^2] - (E[B])^2
\]

\[
= \frac{1}{n} \left( p \left( \frac{1}{n} \right)^2 \right)
\]

\[
= p - p^2 = p (1-p)
\]

\[
\text{Var (Binom) = } n \cdot p (1-p)
\]

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**Chebyshev Inequality**

\[
\Pr( |X - E[X]| \geq K ) \leq \frac{\text{Var}(X)}{K^2}
\]

**Proof**

By Markov’s Inequality

\[
\Pr( X \geq a ) \leq \frac{E[X]}{a}
\]

Put

\[
x \leftarrow (X - E[X])^2
\]

\[
a \leftarrow K^2
\]
\[ \Pr \left( (X - \mathbb{E}[X])^2 > k^2 \right) \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{k^2} \]

\[ \Rightarrow \Pr \left( |X - \mathbb{E}[X]| > k \right) \leq \frac{\text{Var}(X)}{k^2} \]

Useful for bounding sums of R.V.

\[ \text{E.g.: } \Pr \left( x \notin (\mathbb{E}[x] - \text{Var}(x)/\sqrt{2}, \mathbb{E}[x] + \text{Var}(x)/\sqrt{2}) \right) \leq \frac{1}{2} \]

i.e. most of the probability is concentrated around the mean.