Problem 1
For each of the following sequences of integers, find a simple formula or rule which generates the formula. You can either make it a closed form or recursive rule if possible, or explain the pattern/rule in English.

a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...
b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...
c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...
d) 1, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, ...
e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...
f) 1, 3, 15, 105, 945, 10395, 135135, 207025, 34459425, ...
g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, ...
h) 2, 4, 16, 256, 65536, 4294967296, ...

Problem 2

a) Show that for any sequence $a_0, a_1, a_2, ..., a_n$ of real numbers that
$$\sum_{j=1}^n (a_j - a_{j-1}) = a_n - a_0.$$

b) Use the formula above and the fact that
$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$
to compute
$$\sum_{j=1}^n \frac{1}{j(j+1)}.$$

Problem 3 Use Mathematical Induction to prove the following results:

a) $\sum_{k=1}^n j(j + 1)(j + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers $n$. 
b) Any postage amount greater than 7 cents can be formed using a combination of 3-cent and 5-cent stamps.

c) For all integers \( n \geq 0 \), \( 6|(n^3 - n) \).

d) For all integers \( n \geq 4 \), \( n^2 \leq n! \).

e) A three dimensional chessboard of size \( 2^n \times 2^n \times 2^n \) with one 1 x 1 x 1 cube missing can be completely covered by 2 x 2 x 2 cubes, each with one 1 x 1 x 1 cube missing.

**Problem 4**

What is wrong with the following induction proof in which we are trying to prove that for all positive integers \( n \), \( \sum_{i=1}^{n} i = \frac{(n+1)^2}{2} \)?

i) Basis Step: The formula is true for \( n = 1 \).

ii) Assume \( \sum_{i=1}^{k} i = \frac{(k+1)^2}{2} \).

iii) Inductive step: From the assumption, we get \( \sum_{i=1}^{k+1} i = (k + 1) + \left( \frac{(k+\frac{1}{2})^2}{2} \right) + (k + 1) = \frac{k^2 + 3k + \frac{1}{4}}{2} = \frac{(k+\frac{3}{2})^2}{2} = \frac{(k+1)^2 + \frac{1}{4}}{2} \). Hence, the statement is true for all positive integers \( n \).

**Problem 5** A *complete binary tree* is one where there is one node as the root, and each node has exactly 2 "children". A leaf is a node with no children, and the height refers to how deep the tree goes. So, if there is only one node (the root), the tree has height 0. If there is the root and two children, then there are 2 leaves, and the height is 1. If there are 3 levels, the height is 3, and there is 1 node at height 0, 2 at height 1 and 4 at height 2, etc. Prove the following via induction:

a) A complete binary tree of height \( h \) has \( 2^h \) leaves (or there are \( 2^h \) nodes at height \( h \)). b) A complete binary tree of height \( h \) has \( 2^{(h+1)} - 1 \) nodes. Alternatively, you can prove that \( 1 + 2 + 4 + \ldots + 2^h = 2^{(h+1)} - 1 \).

**BONUS Problems**

A) A woman has two friends who live in different parts of the city from her (one is uptown and one is downtown), but they both live off of the same subway line that runs every 10 minutes in both directions, and they are always on time. She decides which friend to visit based on which train comes first, but no matter what times she shows up, 90 percent of the time, she ends up going downtown. Why?

B) You are told that you can climb a ladder one step at a time or two steps at a time, so in order to get to step 2, there are two ways you can proceed: you
can do one big step to step 2 or two small steps of size one. Similarly, there are three ways to get to step 3 (what are they). How many ways can you get to step 100? How about step n?