Chapter 5 - Ray tracing algorithm and geometry.

1 Ray tracing algorithm

We consider the problem of, be given a 3D scene and a point of view in this scene, computing the 2D image of the scene viewed from the given point of view. There exist two families of algorithms to solve this problem: the rasterization algorithms, and the ray tracing algorithms, also called ray casting algorithm.

The goal of this course is two describe a ray tracing algorithm while introducing the main notion of geometry it involves.

1.1 Modelization of the problem

3D scene  A 3D scene can be modeled by a set of triangular patches, that are 3D triangles having each a color. Figure 1 shows a very simple 3D scene consisting in four triangular patches having respective vertices \( (p_1, p_2, p_3) \), \( (p_1, p_2, p_4) \), \( (p_2, p_3, p_4) \) and \( (p_1, p_3, p_4) \).

We will model a 3D scene with \( v \) vertices \( p_1, \ldots, p_v \) and \( m \) patches in Matlab as follows:

- a \( v \times 3 \) array Points which \( i \)-th row is the vector of coordinates of \( p_i \),
- a \( m \times 3 \) array Scene which \( i \)-th row is the 3-tuple containing the indexes in Points of the vertices of the \( i \)-th patch,
- a \( m \times 3 \) array Colors which \( i \)-th row is a triplet describing the color of the \( i \)-th patch.

In Matlab, colors can be handled by RGB triplets, that are three-element row vectors whose elements specify the intensities of the red, green, and blue components of the color; the intensities must be in the range \([0,1]\).

The above 3D scene can be displayed with the commands:
Figure 1: A very simple 3D scene.

```matlab
>> patch('Vertices', Points, 'Faces', Scene, 'FaceVertexCData', Colors, 'FaceColor', 'flat');
>> view(3);
```

The way a 2D image is computed from the 3D scene is the following: we consider that a camera takes a picture of the scene, and the 2D image will be this picture. Here we describe a model for a camera, called the pinhole model, or modèle sténopé in French.

**Pinhole model**  This is illustrated in figure 2.

The main elements of this model are:
• a global reference: a point $o$ and 3 vectors $\vec{x}_o, \vec{y}_o, \vec{z}_o$ forming an orthonormal basis of $\mathbb{R}^3$ (we will suppose that $o = \langle 0, 0, 0 \rangle$ and $\langle \vec{x}_o, \vec{y}_o, \vec{z}_o \rangle$ is the standard basis),

• a camera reference: the position $c$ of the lens, and 3 vectors $\vec{x}_c, \vec{y}_c, \vec{z}_c$ forming an orthonormal basis of $\mathbb{R}^3$, characterizing the orientation of the lens; we will suppose that $\vec{z}_c$ is the direction where the camera looks at; the line directed by $\vec{z}_c$ is called the optical axis,

• a focal length $f$, defining the image plane as the plane normal to $\vec{z}_c$ with distance $f$ to $c$. The intersection of the optical axis with the image plane is the principal point, noted $p$. Note that $p = c + f\vec{z}_c$;

• a width $w$ and an height $h$, defining a rectangular zone centered in $p$ of the image plane. This rectangular zone is called the sensor.

With this model, a 2D image is the central projection with respect to $c$ on the sensor of the visible parts of the 3D scene (see the blue point in figure 2). The visible parts of the scene are the parts that are not hidden by other parts from the given point of view.

Since we are interested in forming digital images, we consider that the sensor is a set of rectangular pixels. We add this to our model:

• the resolution of the sensor is defined by two positive integers $r_w, r_h$ that are the numbers of pixels horizontally and vertically. The sensor is then a rectangular grid of $r_w \times r_h$ pixels end each pixel is a rectangular zone of the sensor with size $(w/r_w) \times (h/r_h)$.

The pixels of the sensor are indexed from (1, 1) to $(r_w, r_h)$, the (1, 1)-th pixel being the bottom left pixel, and the $(r_w, r_h)$-th one being the upper right one. We will note $pix(i, j)$ the center of the $(i, j)$-th pixel.

1.2 The ray tracing algorithm

We define the ray passing through the $(i, j)$-th pixel as the half-line $[c, pix(i, j)]$.

The ray tracing approach proceeds as follows: for each pixel $(i, j)$ of the sensor:

1. compute the intersections of the ray $[c, pix(i, j)]$ with the patches of the scene

2. if there is no such intersection, give to the pixel $(i, j)$ the color white (or the color of the background of the scene);
3. else find the patch for which the intersection with \([c, \text{pix}(i, j)]\) is the closest to \(c\), and give to the pixel \((i, j)\) the color of this patch.

1.3 Computing the intersection between a ray and a patch

In this algorithm, the geometric work lies in the computation of the intersection between a ray and a patch. We specify the procedure \texttt{intersectRayPatch}(i, j, k) in figure 1.3 doing this computation.

\begin{center}
\begin{tabular}{|l|}
\hline
\textbf{procedure:} \texttt{intersectRayPatch}(i, j, k) \\
\textbf{input:} \((i, j)\) is the index of a pixel, \(k\) the index of a patch \\
\textbf{output:} If the ray passing through the \((i, j)\)-th pixel intersects the \(k\)-patch, returns the distance between this intersection and \(c\). Otherwise, returns \(+Inf\). \\
\hline
\end{tabular}
\end{center}

Figure 3: \texttt{intersectRayPatch}

The goal of this course is to introduce the notions of geometry required to write this procedure. The goal of the project associated with this course is to write the \texttt{Matlab} function \texttt{intersectRayPatch} specified in the file \texttt{intersectRayPatch.m}.

Be given a pixel \((i, j)\) and a patch with index \(k\), the main steps we propose to address are:

1. expressing \(\text{pix}(i, j)\) in the global reference,

2. carrying out a system of equations defining the line \(l = (c, \text{pix}(i, j))\),

3. carrying out a system of equations defining the plane \(pl\) passing through the three vertices of the \(k\)-th patch,

4. computing the intersection \(u\) of \(l\) and \(pl\) (we will suppose that \(l\) and \(pl\) are such that this intersection exists),

5. deciding if \(u\) lies in the \(k\)-th patch,

6. deciding if \(u\) lies in the ray \([c, \text{pix}(i, j)]\).

Here, the line \(l\) and the plan \(pl\) we are dealing with do not contain \(\vec{0}\), and are not linear subspaces of \(\mathbb{R}^3\). However, one can see such objects as lines and planes passing through zero.
translated by a vector. These objects are then similar to linear subspaces; they are called **affine subspaces**.

We introduce affine subspaces, coordinate systems and changes of coordinate systems in section 2. In section 3, we describe representations of 3D lines and planes with systems of linear equations.

1.4 Recall: application of solving a syst. of lin. eqns.

Be given a linearily independant family of vectors $B = \langle \vec{b}_1, \ldots, \vec{b}_k \rangle$ and a vector $\vec{v}$, we would like to decide if $\vec{v}$ is in $Span(B)$, and if yes compute its coordinates in $B$.

**Theorem 1** (thm 5.9 of the textbook). Let $B = \langle \vec{b}_1, \ldots, \vec{b}_k \rangle$ be a family of $n$-dimensional vectors, and $\vec{v}$ be a $n$-dimensional vector. Let $B$ be the $n \times k$ matrix which $i$-th column is $\vec{b}_i$, for $1 \leq i \leq k$. Then $\vec{v}$ is in $Span(\langle \vec{b}_1, \ldots, \vec{b}_k \rangle)$ if and only if the system $B\vec{u} = \vec{v}$ has a solution. If it does, then any vector $\vec{u}$ satisfying $B\vec{u} = \vec{v}$ is such that $\vec{v} = \vec{u}[1]\vec{b}_1 + \ldots + \vec{u}[k]\vec{b}_k$.

**Proof of thm.** 1. The proof is in the theorem itself:

- Suppose $\vec{v}$ is in $Span(\langle \vec{b}_1, \ldots, \vec{b}_k \rangle)$: then it exists $\vec{u}$ such that $\vec{v} = \vec{u}[1]\vec{b}_1 + \ldots + \vec{u}[k]\vec{b}_k$ and $B\vec{u} = \vec{v}$ has a solution.

- Suppose $B\vec{u} = \vec{v}$ has a solution. Then this solution is such that $\vec{v} = \vec{u}[1]\vec{b}_1 + \ldots + \vec{u}[k]\vec{b}_k$ and $\vec{v}$ is in $Span(\langle \vec{b}_1, \ldots, \vec{b}_k \rangle)$.

When $B = \langle \vec{b}_1, \ldots, \vec{b}_k \rangle$ is the basis of a subspace of $\mathbb{R}^n$ of dimension $k$, then for any given $\vec{v}$, a solution of $B\vec{u} = \vec{v}$, if it exists, is the **unique** vector $Coords(\vec{u}, B)$.

Actually, the two following cases can happen:

- the system $B\vec{u} = \vec{v}$ is of **Category I**.: $B$ is a square $n \times n$ matrix, $B$ is a basis of $\mathbb{R}^n$ and for any $\vec{v} \in \mathbb{R}^n$, $B\vec{u} = \vec{v}$ admits a unique solution.

- the system $B\vec{u} = \vec{v}$ is of **Category III**.: $k < n$, $Rank(B) = k$ and $Dim(Null(B)) = 0$ and if $\vec{v} \in Image(B)$, then $B\vec{u} = \vec{v}$ has a unique solution. Otherwise ($\vec{v} \in \mathbb{R}^n \setminus Image(B)$), $B\vec{u} = \vec{v}$ has no solution.
Example (Geometry). Recall that the plane $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace $\mathcal{U}$ of $\mathbb{R}^3$ and has $\mathcal{B} = \langle (1, 1, -2), (-2, 1, 1) \rangle$ as a basis.

Consider the vector $\vec{v} = \langle 2, 3, -5 \rangle$. Is it in $\text{Span}(\mathcal{B})$?
Yes, because $B\vec{u} = \vec{v}$ admits the solution $\langle \frac{8}{3}, \frac{1}{3} \rangle$. Since $\mathcal{B}$ is a basis of $\mathcal{U}$, $\langle \frac{8}{3}, \frac{1}{3} \rangle$ are the coordinates of $\vec{v}$ in $\mathcal{B}$.

Consider the vector $\vec{v} = \langle 1, 1, 1 \rangle$. Is it in $\text{Span}(\mathcal{B})$? No, because $B\vec{u} = \vec{v}$ has no solution.
2 Affine subspaces of $\mathbb{R}^n$ - systems of coordinates

2.1 Affine subspaces of $\mathbb{R}^n$

We call points, and note it with lowercase letters like $a$, the elements of the set $\mathbb{R}^n$. The coordinates of points are noted as $n$-tuples: $a = (a_1, \ldots, a_n)$.

We call arrows, and note it with lowercase letters with an over-right arrow, as vectors, the elements of the vector space $\mathbb{R}^n$.

Be given two points $a, b \in \mathbb{R}^n$ and two arrows $\vec{u}, \vec{v} \in \mathbb{R}^n$, we define:

- $a - b$ as the arrow $\langle a[1] - b[1], \ldots, a[n] - b[n] \rangle$,
- $a + \vec{u}$ as the point $\langle a[1] + \vec{u}[1], \ldots, a[n] + \vec{u}[n] \rangle$,
- $\vec{u} + \vec{v}$ as the arrow $\langle \vec{u}[1] + \vec{v}[1], \ldots, \vec{u}[n] + \vec{v}[n] \rangle$.

**Definition 1** (Affine subspace of $\mathbb{R}^n$). A subset set $U$ of $\mathbb{R}^n$ is called an affine subspace of $\mathbb{R}^n$ if it exists a (linear) subspace $U$ of $\mathbb{R}^n$ and a point $u \in \mathbb{R}^n$ such that any element of $U$ can be written as the sum of $u$ with an arrow of $U$. That is, $\forall a \in U$, $\exists \vec{u} \in U$ s.t. $a = u + \vec{u}$.

**Definition 2** (Dimension of an affine subspace). Let $U$ be an affine subspace of $\mathbb{R}^n$ associated with the subspace $U$. We call $\text{Dim}(U)$ the dimension of $U$, and we note it $\text{Dim}(U)$.

**Example.** in $\mathbb{R}^3$:

- $\mathbb{R}^3$ is an affine subspace of $\mathbb{R}^3$, associated with the subspace $\mathbb{R}^3$ of $\mathbb{R}^3$: any point $p = \langle p_1, p_2, p_3 \rangle \in \mathbb{R}^3$ can be written $\langle 0, 0, 0 \rangle + \vec{p}$ where $\vec{p} = \langle p_1, p_2, p_3 \rangle$.
- Let $U$ be a subspace of $\mathbb{R}^3$ and $U$ be the subset of $\mathbb{R}^3$ defined by $\{\langle 0, 0, 0 \rangle + \vec{u} | \vec{u} \in U\}$. $U$ is an affine subspace of $\mathbb{R}^3$.
- Let $a, b$ two different points of $\mathbb{R}^3$.
  - $L = \{t \ast (b - a) | t \in \mathbb{R}\}$ is a 1-dimensional subspace of $\mathbb{R}^3$ having $\langle (b - a) \rangle$ as a basis.
  - $L = \{a + \vec{u} | \vec{u} \in L\} = \{a + t \ast (b - a) | t \in \mathbb{R}\}$ is a 1-dimensional affine subspace of $\mathbb{R}^3$. It is the line $(a, b)$ passing through $a$ and $b$.
- Let $a, b, c$ be 3 distinct points of $\mathbb{R}^3$ that do not lie on the same line.
  - $P = \{s \ast (b - a) + t \ast (c - a) | s, t \in \mathbb{R}\}$ is a 2-dimensional subspace of $\mathbb{R}^3$ having
\[(b - a), (c - a)\) as a basis.

\[P = \{a + \bar{u}|\bar{u} \in \mathcal{P}\} = \{a + s*(b - a) + t*(c - a)|s \in \mathbb{R}, t \in \mathbb{R}\}\] is a 2-dimensional affine subspace of \(\mathbb{R}^3\). It is the plane passing through \(a, b\) and \(c\).

### 2.2 Systems of coordinates

**Definition 3 (System of coordinates).** Let \(U\) be an affine subspace of \(\mathbb{R}^n\) of associated with the subspace \(U\). A system of coordinates for \(U\), is a couple \(C = (u, B_U)\) where \(u\) is a point of \(U\), and \(B_U\) is a basis of \(U\).

If \(B_U\) is orthogonal, \(C = (u, B_U)\) is said orthogonal. If \(B_U\) is orthonormal, \(C = (u, B_U)\) is said orthonormal.

Then, we have the following:

**Proposition 1.** Let \(U\) be an affine subspace of dimension \(m\) of \(\mathbb{R}^n\) associated with the subspace \(U\) of \(\mathbb{R}^n\), and \(C = (u, B_U)\) be a coordinate system for \(U\) where \(B_U = \langle \vec{b}_1, \ldots, \vec{b}_m \rangle\).

For any point \(a \in U\) there exists a unique tuple \(\langle a_1, \ldots, a_m \rangle\) noted \(\text{Coords}(a, C)\) and called coordinates of \(a\) in \(C\), such that \(a = u + a_1\vec{b}_1 + a_2\vec{b}_2 + \ldots + a_m\vec{b}_m\).

**Example.** in \(\mathbb{R}^3\):

- \(\mathcal{S} = ((0, 0, 0), \langle \vec{e}^x, \vec{e}^y, \vec{e}^z \rangle)\) where \(\langle \vec{e}^x, \vec{e}^y, \vec{e}^z \rangle\) is the standard basis of \(\mathbb{R}^3\) is an orthonormal coordinate system of \(\mathbb{R}^3\). We will call it the standard coordinate system or \(\mathbb{R}^3\).

- \(\mathcal{S} = ((0, \ldots, 0), \langle \vec{e}^1, \ldots, \vec{e}^n \rangle)\) where \(\langle \vec{e}^1, \ldots, \vec{e}^n \rangle\) is the standard basis of \(\mathbb{R}^n\) is an orthonormal coordinate system of \(\mathbb{R}^n\). We will call it the standard coordinate system or \(\mathbb{R}^n\).

- \(\mathcal{C}_L = (a, \langle b - a \rangle)\) is a system of coordinates for the line \(L\) defined above. Let \(t \in \mathbb{R}\) and \(u = a + t*(b - a)\) be a point of \(L\). Then \(\text{Coords}(u, \mathcal{C}_L) = \langle t \rangle\).

- \(\mathcal{C}_P = (a, \langle b - a, c - a \rangle)\) is a system of coordinates for the plane \(P\) defined above. Let \(s \in \mathbb{R}, t \in \mathbb{R}\) and \(u = a + t*(b - a) + s*(c - a)\) be a point of \(P\). Then \(\text{Coords}(u, \mathcal{C}_P) = \langle s, t \rangle\).

### 2.3 Changing coordinate systems

Let \(U\) be an affine subspace of dimension \(m\) of \(\mathbb{R}^n\) associated with the subspace \(U\) of \(\mathbb{R}^n\), and \(C = (u, B_U)\) a coordinate system for \(A\) where \(B_U = \langle \vec{b}_1, \ldots, \vec{b}_m \rangle\). Let \(a\) be a point of \(U\). We
focus now on how passing from one coordinate system to another.

**From Coords\((a,C)\) to Coords\((a,S)\):** Let \(\text{Coords}(a,C) = \langle a_1, \ldots, a_m \rangle\) be the coordinates of \(a\) in \(C\). As highlighted in proposition 1, one has \(a = u + a_1 \vec{b}_1 + a_2 \vec{b}_2 + \ldots + a_m \vec{b}_m\); i.e. the coordinates of \(a\) in the standard coordinate system are \(\text{Coords}(a,S) = u + a_1 \vec{b}_1 + a_2 \vec{b}_2 + \ldots + a_m \vec{b}_m\).

**From Coords\((a,S)\) to Coords\((a,C)\):** Remark first that if \(a \in U\), then \(a - u\) is an arrow of \(U\). As a consequence, \(\text{Coords}(a - u, B_U)\) is well defined. Then, i.e. \(\text{Coords}(a,C) = \text{Coords}(a - u, B_U)\).

Then (from theorem 5.9, page 120 of the textbook or thm. 1 of this document) \(\text{Coords}(a - u, B_U)\) is the unique solution of the system of linear equations \(M\vec{x} = \vec{v}\), where \(\vec{v}\) is the vector \(a - u = \text{Coords}(a,S) - \text{Coords}(u,S)\) and \(M\) is the matrix which columns are \(\vec{b}_1, \ldots, \vec{b}_m\).
3 Lines and planes in $\mathbb{R}^3$

Let us enumerate the affine subspaces of $\mathbb{R}^3$ with respect to their dimensions. Let $U = \{ u + \bar{u} | \bar{u} \in \mathcal{U} \}$ be an affine subspace and suppose it has dimension:

0: $\mathcal{U}$ has dimension 0 and is $\{ \vec{0} \};$ $U$ is the point $u.$

1: $\mathcal{U}$ contains all the arrows colinear to a given arrow $\vec{u};$ $U$ is the line passing through $u$ directed by the arrow $\vec{u}.$

2: $\mathcal{U}$ contains all the linear sums of 2 non-colinear arrows, it is a plane passing through 0; $U$ is a plane parallel to $\mathcal{U},$ passing through $u.$

3: $\mathcal{U}$ has dimension 3 and is $\mathbb{R}^3; U$ is the whole space $\mathbb{R}^3.$

For the sake of consistency, we define the left multiplication of a point $u \in \mathbb{R}^n$ by a $m \times n$ matrix $A,$ noted $Au,$ as the $m$ dimensional arrow $A(u - \langle 0, \ldots, 0 \rangle).$

**Proposition 2.** Let $U = \{ u + \bar{u} | \bar{u} \in \mathcal{U} \}$ be a subspace of $\mathbb{R}^n$ of dimension $m.$ Then it exists a $(n - m) \times n$ matrix $A$ of rank $n - m$ and $\vec{w} \in \mathbb{R}^n$ such that $U = \{ v \in \mathbb{R}^n | Av = \vec{w} \}.$

Reciprocally, let $A$ be an $l \times n$ matrix of rank $k \leq l \leq n,$ and $\vec{w} \in \mathbb{R}^l.$ Then the set $U = \{ v \in \mathbb{R}^n | Av = \vec{w} \}$ is an affine subspace of dimension $n - k.$

**Proof of prop. 2:** Reciprocal: consider first the subspace $\text{Null}(A)$ of $\mathbb{R}^n$ defined by $A\vec{v} = \vec{0}.$ From the rank nullity theorem, it has dimension $n - k.$ Let $u$ be a solution of $Av = \vec{w}.$ For any $\vec{u} \in \text{Null}(A),$ $A(u + \vec{u}) = Au + A\vec{u} = \vec{w},$ and $U = \{ u + \vec{u} | \vec{u} \in \text{Null}(A) \}$ is an affine subspace of dimension $n - k.$

First part of the prop: Let $U = \{ u + \bar{u} | \bar{u} \in \mathcal{U} \}$ be a subspace of $\mathbb{R}^n$ of dimension $m.$ Then $\mathcal{U}$ is a subspace of $\mathbb{R}^n$ of dimension $m,$ and has an orthogonal complement $\mathcal{V}$ in $\mathbb{R}^n$ and $\mathcal{V}$ has dimension $n - m$ (see theorem 4.23 of the textbook, page 89). Let $\langle \vec{v}_1, \ldots, \vec{v}_{n-m} \rangle$ be any basis of $\mathcal{V}$ and $A$ be the matrix which row $A[i, :]$ is $\vec{v}_i.$ For any arrow $\vec{u} \in \mathcal{U},$ $A\vec{u} = \vec{0}.$ $A$ has size $(n - m) \times n$ and has rank $n - m$ because $\vec{v}_1, \ldots, \vec{v}_{n-m}$ are linearly independent. Finally, one can easily show that $U$ is the set of solutions of the system $Av = Au$ where $v$ is the unknown. ∎
3.1 Lines of $\mathbb{R}^3$

One can give three different representations of a line in $\mathbb{R}^3$:

**Representation in term of points:** A line $L$ in $\mathbb{R}^3$ is fully defined by two different points $a, b$ of $\mathbb{R}^3$, and we note $(a, b)$ the unique line passing through $a$ and $b$.

**Parameterized representation:** Since the line $L$ is an affine subspace of $\mathbb{R}^3$, it can be written $L = \{u + t\vec{u} | t \in \mathbb{R}\}$ where $L$ is a linear subspace of $\mathbb{R}^3$ of dimension 1. Then it exists $\vec{u} \in \mathbb{R}^3$ s.t. $L = \{u + t\vec{u} | t \in \mathbb{R}\}$, and we call this a parameterized representation of $L$.

Suppose $L$ passes through two different points $a, b$ of $\mathbb{R}^3$. Then a parameterized representation is $L = \{a + t(b - a) | t \in \mathbb{R}\}$, that is $L = \text{Span}(\langle b - a \rangle)$, and $\vec{u} = a$.

We now consider the inverse problem, that is be given a parameterized representation $\{u + t\vec{u} | t \in \mathbb{R}\}$ of the line $L$, find its representation in term of points. This is easily solved: $L$ is the line passing through $u$ and $u + \vec{u}$.

**Implicit representation:** (called “linear equations” in the textbook)

Let $L = \{a + t(b - a) | t \in \mathbb{R}\}$. According to prop. 2 $L$ is the set of solutions of a system of linear equations $Av = \vec{w}$ where $A$ is a $2 \times 3$ matrix of rank 2 and $\vec{w} \in \mathbb{R}^2$.

According to the proof of the first part of prop. 2 if $\langle \vec{v}_1, \vec{v}_2 \rangle$ is the basis of an orthogonal complement of $\text{Span}(\langle b - a \rangle)$, $A$ is the matrix which rows are $\vec{v}_1$ and $\vec{v}_2$, and $\vec{w}$ is $Aa$.

We now consider the inverse problem, that is be given an implicit representation $Av = \vec{w}$ of $L$, compute a parameterized representation $\{u + t\vec{u} | t \in \mathbb{R}\}$ of $L$. According to the proof of the second part of prop. 2 if $\langle \vec{u} \rangle$ is a basis of $\text{Null}(A)$, and $u$ is any solution of $Av = \vec{w}$, $\{u + t\vec{u} | t \in \mathbb{R}\}$ is a parameterized representation of $L$.

3.2 Planes of $\mathbb{R}^3$

**Representation in term of points:** A plane $P$ in $\mathbb{R}^3$ is fully defined by three pairwise distinct points $a, b, c$ of $\mathbb{R}^3$ that are not on the same line.

**Parameterized representation:** The plane $P$ is an affine subspace of $\mathbb{R}^3$ of dimension 2 and can be written $P = \{u + t\vec{u} | \vec{u} \in \mathcal{P}\}$ where $\mathcal{P}$ is a linear subspace of $\mathbb{R}^3$ of dimension 2. A
parameterized representation has the form $P = \{a + s\vec{u}_1 + t\vec{u}_2 | s \in \mathbb{R}, t \in \mathbb{R}\}$ where $\langle \vec{u}_1, \vec{u}_2 \rangle$ is a basis of $\mathcal{P}$.

Suppose $P$ passes through $a, b$ and $c$ that are distinct and not on the same line. Then $b - a$ and $c - a$ are two independent arrows of $\mathcal{P}$ and $\langle b - a, c - a \rangle$ is a basis of $\mathcal{P}$. Then $P = \{a + s(b - a) + t(c - a) | s \in \mathbb{R}, t \in \mathbb{R}\}$ is a parameterized representation of $P$.

Inverse problem: let $\{u + s\vec{u}_1 + t\vec{u}_2 | s \in \mathbb{R}, t \in \mathbb{R}\}$ be a parameterized representation of $P$. $P$ is the plane passing through $u, u + \vec{u}_1, u + \vec{u}_2$.

**Implicit representation:** Let $P = \{a + s(b - a) + t(c - a) | s \in \mathbb{R}, t \in \mathbb{R}\}$, and $\vec{v}$ be a vector orthogonal to both $(b - a)$ and $(c - a)$ (i.e. $\text{Span}(\langle \vec{v} \rangle)$ is an orthogonal complement to $\text{Span}(\langle b - a, c - a \rangle)$). If $A$ is the $1 \times 3$ matrix which row is $\vec{v}$, $P$ is the set of solutions of $A\vec{u} = Aa$.

There is a “trick” for finding a basis for the orthogonal complement of a plan; this trick is the cross-product. The cross product is an operation defined for two arrows $\vec{u} = \langle \vec{u}[1], \vec{u}[2], \vec{u}[3] \rangle$ and $\vec{v} = \langle \vec{v}[1], \vec{v}[2], \vec{v}[3] \rangle$ of $\mathbb{R}^3$ as the arrow of $\mathbb{R}^3$:

$$\vec{u} \times \vec{v} = \langle \vec{u}[2] \times \vec{v}[3] - \vec{u}[3] \times \vec{v}[2], \vec{u}[3] \times \vec{v}[1] - \vec{u}[1] \times \vec{v}[3], \vec{u}[1] \times \vec{v}[2] - \vec{u}[2] \times \vec{v}[1], \rangle.$$ 

If $\vec{u}$ and $\vec{v}$ are not colinear, then $\vec{u} \times \vec{v} \neq \vec{0}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. 
