Problem Set 3

Assigned: June 13
Due: June 27

Problem 1

An $n \times n$ matrix A is *nilpotent* if $A^p = 0$ for some exponent p.

A. A matrix is *strictly upper triangular* if all the non-zero elements lie above the main diagonal; that is, if $j \leq i$ then $A[i,j] = 0$. For instance, the matrix

$$\begin{bmatrix} 0 & 4 & 3 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is upper triangular. Show that any upper triangular matrix is nilpotent. (Hint: Compute the successive powers of the above matrix and notice what happens. Prove that that has to happen in general.)

B. Show that if A is nilpotent and C is a non-singular matrix, then $C^{-1}AC$ is nilpotent.

C. Construct a 4×4 nilpotent matrix A that has no zero elements. Hint: Use (a) part B; (b) the fact that almost all $n \times n$ matrices are non-singular; (c) Matlab.

D. Suppose that A is a nilpotent matrix; that vector $\vec{v} \neq \vec{0}$; and that constant $c \neq 0$. Show that $A \cdot \vec{v} \neq c \cdot \vec{v}$. (In technical terminology, 0 is the only eigenvalue of A.)

Problem 2

This problem depends on Programming Assignment 2, problem 1.

A. Give an argument to show that, if k is even, then in general there does not exist a solution to the inverse problem. (Hint: Show that the last row of the coefficient matrix C can be written as a linear sum of the other rows. Explain why this proves the result.)

B. Give an argument to show that, if k is odd, then there always exists a solution to the inverse problem. (Hint: In MATLAB, construct the coefficient matrix C for some small odd k — e.g. $k = 5$ or $k = 7$ — and compute its inverse. You will see that the inverse has a very simple, regular pattern. Argue that you can use this pattern to construct the inverse of C for any odd value of k.)

For problems 3-5 you may use Matlab. You should show the computation involved, not just give the answer.

Problem 3

Let P be the plane in $\mathbb{R}^3$ containing the three points $\langle 1,0,1\rangle$ $\langle 1,2,3\rangle$ and $\langle -1,2,1\rangle$ and let Q be the plane containing the three points $\langle 5,2,1\rangle$, $\langle 1,0,1\rangle$, and $\langle -3,-1,-1\rangle$. A. Find the line that is the intersection of P and Q and represent it in parameterized form $\{p + t \cdot \vec{v} \,|t \in \mathbb{R}\}$

B. Find the point on the line in A that is closest to the point $\langle 2,3,1\rangle$

C. Find the plane that is perpendicular to the line in (A) and contains the point in (B).

Problem 4

Find the distance between the line $\langle 1,0,2\rangle + t \cdot \langle 1,-1,1\rangle$ and the line $\langle 3,-2,-1\rangle + u \cdot \langle -2,1,1\rangle$. Hint: If L and M are two skew lines in $\mathbb{R}^3$ (or higher dimension) — that is, two lines that are not parallel but do not intersect — and $\mathbf{p}$ and $\mathbf{q}$ are their two closest points, then the line containing $\mathbf{p}$ and $\mathbf{q}$ is perpendicular to both L and M.

Problem 5

Let P be the same plane as in Problem 3. Are the two points $\langle 12,16,-2\rangle$ and $\langle -5,18,-10\rangle$ on the same side of P or on opposite sides?