Problem Set 2

Assigned: May 30
Due: June 13

Problem 1

For each of the following sets of vectors, state whether it is linearly dependent or linearly independent and explain your answer in a sentence. You should refer to theorems in the book when they are relevant. You should be able to do these by inspection, without setting pencil to paper, let alone starting up MATLAB.

A. $\{ \langle 1, 7, 3 \rangle,$
$\langle 4, 2, 4 \rangle$
$\langle 0, 0, 0 \rangle$
$\}$

B. $\{ \langle 3, 0, 0, 0 \rangle,$
$\langle 3, 2, 0, 0 \rangle,$
$\langle 2, 0, -2, 0 \rangle$
$\}$

C. $\{ \langle 4, 2, 2 \rangle,$
$\langle 3, -1, 1 \rangle,$
$\langle 1, 1, -2 \rangle,$
$\langle 2, 4, 5 \rangle$
$\}$

D. $\{ \langle 1, 2, 4 \rangle,$
$\langle 1, 7, 2 \rangle$
$\}$

Problem 2

Suppose you have two sets of k n-dimensional vectors; e.g. ($k = 3$, $n = 7$)

$$\begin{aligned}
\mathcal{P} = \{ & \langle 1, 2, 3, 4, 5, 6, 7 \rangle, \\
& \langle 2, -3, 5, -8, 13, -21, 34 \rangle, \\
& \langle 3, 5, 7, 9, -11, -13, -15 \rangle, \\
\} & \\
\mathcal{Q} = \{ & \langle 1, 3, 2, 5, 4, 7, 6 \rangle, \\
& \langle 1, -1, 2, -2, 3, 4, -5 \rangle, \\
& \langle 1, 0, 4, 0, 2, 0, 8 \rangle \\
\} &
\end{aligned}$$

How can you easily determine in MATLAB determine whether these span the same subspace; i.e. is $\mathrm{Span}(\mathcal{P}) = \mathrm{Span}(\mathcal{Q})$?

Problem 3

A *n-dimentional permutation* is a function over an n-dimensional vectors that reorders their elements in a fixed way. For instance, one 6-dimensional permutation σ maps any 6-dimensional vector $\vec{v} = [a, b, c, d, e, f]$ to the vector $\sigma(\vec{v}) = [c, f, a, b, e, d]$.

A *permutation matrix* is a square matrix such that each row and each column has one entry of 1 and the rest 0. For instance, the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is a 6×6 permutation matrix.

As you can easily check, for any 6-dimensional vector $\vec{v}$, $\sigma(\vec{v}) = A \cdot v$, where σ and A are the permutation and matrix given above.

A. Show that a permutation is a linear transformation.

B. Consider the function $\text{sort}(\vec{v})$, which returns the elements of $\vec{v}$ in sorted order. Give an example to show that $\text{sort}(\vec{v})$ is not a linear transformation.

C. Show that if A is any $n \times n$ permutation matrix, then there exists a permutation σ such that for any vector $\vec{v}$, $\sigma(\vec{v}) = A \cdot \vec{v}$. Show that if σ is any permutation, then there exists a permutation matrix A such that $A \cdot \vec{v} = \sigma(\vec{v})$. Describe how A can easily be computed from σ, and vice versa.

D. Show that, for any permutation matrix A, $A^T \cdot A = I_n$. Therefore $A^{-1} = A^T$.

E. Show that, for any two permutation matrices A and B, $A \cdot B$ is also a permutation matrix.

In all parts of this question except (B), "Show that" means "Give an argument or proof that this statement always holds". It does *not* mean "Give one particular example."