Discrete Mathematics
Lecture 2
Continuing Logic, Quantified Logic, Beginning Proofs

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Administration

- Class Web Site

  http://cs.nyu.edu/courses/summer16/CSCI-GA.2340-001/
- Mailing List
  Subscribe at see website

  Messages to: see website
- TA/Office Hours, etc: Jianbo Sun
- Homework

Conditional Statements

- Write truth table for: $p \wedge q \rightarrow \sim p$
- Show that $(p \vee q) \rightarrow r = (p \rightarrow r) \wedge (q \rightarrow r)$
- Representation of $\rightarrow$: $p \rightarrow q = \sim p \vee q$
- Re-write using if-else: Either you get in class on time, or you risk missing some material
- Negation of $\rightarrow$: $\sim(p \rightarrow q) = p \wedge \sim q$
- Write negation for: If it is raining, then I cannot go to the beach

Conditional Statements

- Contrapositive $p \rightarrow q$ is another conditional statement $\sim q \rightarrow \sim p$
- A conditional statement is equivalent to its contrapositive
- The converse of $p \rightarrow q$ is $q \rightarrow p$
- The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$
- Conditional statement and its converse are not equivalent
- Conditional statement and its inverse are not equivalent

Conditional Statements

- The converse and the inverse of a conditional statement are equivalent to each other
- p only if q means $\sim q \rightarrow \sim p$, or $p \rightarrow q$
- Biconditional of p and q means “p if and only if q” and is denoted as $p \leftrightarrow q$
- r is a sufficient condition for s means “if r then s”
- r is a necessary condition for s means “if not r then not s”

Exercises

- Write contrapositive, converse and inverse statements for:
  - If P is a square, then P is a rectangle
  - If today is Thanksgiving, then tomorrow is Friday
  - If c is rational, then the decimal expansion of r is repeating
  - If n is prime, then n is odd or n is 2
  - If x is nonnegative, then x is positive or x is 0
  - If Tom is Ann's father, then Jim is her uncle and Sue is her aunt
  - If n is divisible by 6, then n is divisible by 2 and n is divisible by 3

Arguments

- An argument is a sequence of statements. All statements except the final one are called premises (or assumptions or hypotheses). The final statement is called the conclusion.
- An argument is considered valid if from the truth of all premises, the conclusion must also be true.
- The conclusion is said to be inferred or deduced from the truth of the premises

Arguments

- Test to determine the validity of the argument:
  - Identify the premises and conclusion of the argument
  - Construct the truth table for all premises and the conclusion
  - Find critical rows in which all the premises are true
  - If the conclusion is true in all critical rows then the argument is valid, otherwise it is invalid
- Example of valid argument form:
  - Premises: $p \vee (q \vee r)$ and $\sim r$, conclusion: $p \vee q$
- Example of invalid argument form:
  - Premises: $p \rightarrow q \vee \sim r$ and $q \rightarrow p \wedge r$, conclusion: $p \rightarrow r$

Valid Argument-Forms

- Modus ponens (method of affirming):
  - Premises: $p \rightarrow q$ and p, conclusion: q
- Modus tollens (method of denying):
  - Premises: $p \rightarrow q$ and $\sim q$, conclusion: $\sim p$
- Disjunctive addition:
  - Premises: p, conclusion: $p \mid q$
  - Premises: q, conclusion: $p \mid q$
- Conjunctive simplification:
  - Premises: $p \,\&\, q$, conclusion: p, q

Valid Argument-Forms

- Disjunctive Syllogism:
  - Premises: $p \mid q$ and $\sim q$, conclusion: p
  - Premises: $p \mid q$ and $\sim p$, conclusion: q
- Hypothetical Syllogism
  - Premises: $p \rightarrow q$ and $q \rightarrow r$, conclusion: $p \rightarrow r$
- Dilemma: proof by division into cases:
  - Premises: $p \mid q$ and $p \rightarrow r$ and $q \rightarrow r$, conclusion: r

Complex Deduction

- Premises:
  - If my glasses are on the kitchen table, then I saw them at breakfast
  - I was reading the newspaper in the living room or I was reading the newspaper in the kitchen
  - If I was reading the newspaper in the living room, then my glasses are on the coffee table
  - I did not see my glasses at breakfast
  - If I was reading my book in bed, then my glasses are on the bed table
  - If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table
- Where are the glasses?

Fallacies

- A fallacy is an error in reasoning that results in an invalid argument
- Three common fallacies:
  - Vague or ambiguous premises
  - Begging the question (assuming what is to be proved)
  - Jumping to conclusions without adequate grounds
- Converse Error:
  - Premises: $p \rightarrow q$ and q, conclusion: p
- Inverse Error:
  - Premises: $p \rightarrow q$ and $\sim p$, conclusion: $\sim q$

Fallacies

- It is possible for a valid argument to have false conclusion and for an invalid argument to have a true conclusion:
  - Premises: if John Lennon was a rock star, then John Lennon had red hair, John Lennon was a rock star; Conclusion: John Lennon had red hair
  - Premises: If New York is a big city, then New York has tall buildings, New York has tall buildings; Conclusion: New York is a big city

Contradiction

- Contradiction rule: if one can show that the supposition that a statement p is false leads to a contradiction , then p is true.
- Knight is a person who always says truth, knave is a person who always lies:
  - A says: B is a knight
  - B says: A and I are of opposite types

  What are A and B?

Digital Logic Circuits

- Digital Logic Circuit is a basic electronic component of a digital system
- Values of digital signals are 0 or 1 (bits)
- Black Box is specified by the signal input/output table
- Three gates: NOT-gate, AND-gate, OR-gate
- Combinational circuit is a combination of logical gates
- Combinational circuit always correspond to some boolean expression, such that input/output table of a table and a truth table of the expression are identical

Number Systems

- Decimal number system
- Binary number system
- Conversion between decimal and binary numbers
- Binary addition and subtraction

Logic of Quantified Statements

Predicates

- A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables
- The domain of a predicate variable is a set of all values that may be substituted in place of the variable
- P(x): x is a student at NYU

Predicates

- If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is the set of all elements in D that make $P(x)$ true when substituted for x. The truth set is denoted as:

  $\{x \in D \mid P(x)\}$

- Let $P(x)$ and $Q(x)$ be predicates with the common domain D. $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$. $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets

Universal Quantifier

- Let P(x) be a predicate with domain D. A universal statement is a statement in the form "$\forall x \in D$, P(x)". It is true iff P(x) is true for every x from D. It is false iff P(x) is false for at least one x from D. A value of x from which P(x) is false is called a counterexample to the universal statement
- Examples
  - $D = \{1, 2, 3, 4, 5\}$: $\forall x \in D, x^2 >= x$
  - $\forall x \in R, x^2 >= x$
- Method of exhaustion

Existential Quantifier

- Let P(x) be a predicate with domain D. An existential statement is a statement in the form "$\exists x \in D, P(x)$". It is true iff P(x) is true for at least one x from D. It is false iff P(x) is false for every x from D.
- Examples:
  - $\exists m \in Z, m^2 = m$
  - $E = \{5, 6, 7, 8, 9\}, \exists x \in E, m^2 = m$

Universal Conditional Statement

- Universal conditional statement "$\forall x$, if $P(x)$ then $Q(x)$":
  - $\forall x$ R, if $x > 2$, then $x^2 > 4$
- Writing Conditional Statements Formally
- Universal conditional statement is called vacuously true or true by default iff $P(x)$ is false for every x in D

Negation of Quantified Statements

- The negation of a universally quantified statement $\forall x \in D, P(x)$ is $\exists x \in D, \sim P(x)$
- "All balls in the bowl are red" – Vacuosly True Example for Universal Statements
- The negation of an existentially quantified statement $\exists x \in D, P(x)$ is $\forall x \in D, \sim P(x)$
- The negation of a universal conditional statement $\forall x \in D, P(x) \rightarrow Q(x)$ is $\exists x \in D, P(x) \wedge \sim Q(x)$

Exercises

- Write negations for each of the following statements:
  - All dinosaurs are extinct
  - No irrational numbers are integers
  - Some exercises have answers
  - All COBOL programs have at least 20 lines
  - The sum of any two even integers is even
  - The square of any even integer is even
- Let P(x) be some predicate defined for all real numbers x, let:

  $r = \forall x \in Z, P(x)$; $s = \forall x \in Q, P(x)$; $t = \forall x \in R, P(x)$
  - Find P(x) (but not $x \in Z$) so that r is true, but s and t are false
  - Find P(x) so that both r and s are true, but t is false

Variants of Conditionals

- Contrapositive
- Converse
- Inverse
- Generalization of relationships from before
- Examples

Necessary and Sufficient Conditions, Only If

- $\forall x$, r(x) is a sufficient condition for s(x) means: $\forall x$, if r(x) then s(x)
- $\forall x$, r(x) is a necessary condition for s(x) means: $\forall x$, if s(x) then r(x) (or $\forall x$, if ~r(x) then ~s(x), if r(x) does not happen, s(x) cannot happen)
- $\forall x$, r(x) only if s(x) means: $\forall x$, if r(x) then s(x) (or $\forall x$, if ~s(x) then ~r (x))

Multiply Quantified Statements

- For all positive numbers x, there exists number y such that y < x
- There exists number x such that for all positive numbers y, y < x
- For all people x there exists person y such that x loves y
- There exists person x such that for all people y, x loves y
- Definition of mathematical limit:

$$\lim_{x \to c} f(x) = L \iff (\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall x \in D)(0 < |x - c| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon)$$

- Order of quantifiers matters in some (most) cases (will find page reference, “lively discussion”: http://math.stackexchange.com/questions/201051/is-the-order-of-universal-existential-quantifiers-important

Negation of Multiply Quantified Statements

- The negation of $\forall x, \exists y, P(x, y)$
  is logically equivalent to $\exists x, \forall y, \sim P(x, y)$
- The negation of $\exists x, \forall y, P(x, y)$
  is logically equivalent to $\forall x, \exists y, \sim P(x, y)$

Prolog Programming Language

- Can use parts of logic as programming lang.
- Simple statements:

  isabove(g, b), color(g, gray)
- Quantified statements:

  if isabove(X, Y) and isabove(Y, Z) then isabove(X, Z)
- Questions:

  ?color(b, blue), ?isabove(X, w)

Exercises

- Determine whether a pair of quantified statements have the same truth values
  - $\forall x \in D, (P(x) \wedge Q(x))$ vs $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
  - $\exists x \in D, (P(x) \wedge Q(x))$ vs $(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$
  - $\forall x \in D, (P(x) \vee Q(x))$ vs $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$
  - $\exists x \in D, (P(x) \vee Q(x))$ vs $(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

Arguments with Quantified Statements

- Rule of universal instantiation: if some property is true of everything in the domain, then this property is true for any subset in the domain
- Universal Modus Ponens:
  - Premises: ($\forall$x, if P(x) then Q(x)); P(a) for some a
  - Conclusion: Q(a)
- Universal Modus Tollens:
  - Premises: ($\forall$x, if P(x) then Q(x)); ~Q(a) for some a
  - Conclusion: ~P(a)
- Converse and inverse errors

Validity of Arguments using Diagrams

- Premises: All human beings are mortal; Zeus is not mortal. Conclusion: Zeus is not a human being
- Premises: All human beings are mortal; Felix is mortal. Conclusion: Felix is a human being
- Premises: No polynomial functions have horizontal asymptotes; This function has a horizontal asymptote. Conclusion: This function is not a polynomial

Proof and Counterexample

- Discovery and proof
- Even and odd numbers
  - number n from Z is called even if $\exists k \in Z, n = 2k$
  - number n from Z is called odd if $\exists k \in Z, n = 2k + 1$
- Prime and composite numbers
  - number n from Z is called prime if

    $\forall r, s \in Z, n = r * s \rightarrow r = 1 \vee s = 1$
  - number n from Z is called composite if

    $\exists r, s \in Z, n = r * s \wedge r > 1 \wedge s > 1$

Proving Statements

- Constructive proofs for existential statements
- Example: Show that there is a prime number that can be written as a sum of two perfect squares
- Universal statements: method of exhaustion and generalized proof
- Direct Proof:
  - Express the statement in the form: $\forall x \in D, P(x) \rightarrow Q(x)$
  - Take an arbitrary x from D so that P(x) is true
  - Show that Q(x) is true based on previous axioms, theorems, P(x) and rules of valid reasoning

Proof

- Show that if the sum of any two integers is even, then so is their difference
- Common mistakes in a proof
  - Arguing from example
  - Using the same symbol for different variables
  - Jumping to a conclusion
  - Begging the question

Counterexample

- To show that the statement in the form "$\forall x \in D$, $P(x) \rightarrow Q(x)$" is not true one needs to show that the negation, which has a form "$\exists x \in D$, $P(x) \wedge \sim Q(x)$" is true. **x** is called a counterexample.
- Famous conjectures:
  - Fermat big theorem: there are no non-zero integers x, y, z such that $x^n + y^n = z^n$, for $n > 2$
  - Goldbach conjecture: any even integer can be represented as a sum of two prime numbers
  - Euler's conjecture: no three perfect fourth powers add up to another perfect fourth power

Exercises

- Any product of four consecutive positive integers is one less than a perfect square
- To check that an integer is a prime it is sufficient to check that n is not divisible by any prime less than or equal to $\sqrt{n}$
- If p is a prime, is $2^p - 1$ a prime too?
- Does $15x^3 + 7x^2 - 8x - 27$ have an integer zero?