

HOMEWORK 3 - DISCRETE MATH - DUE 07/21/16

Assigned: 07/21/16

Due: 08/02/16

Please make sure to clearly write your name at the top of your hand-in. Also, **indicate if you worked with anybody and also indicate how many hours total you worked on the homework**. Feel free to discuss any problems (including the bonuses) on the class mailing list. I am also required to remind all students of the academic integrity policy at http://www.cs.nyu.edu/web/Academic/Graduate/academic_integrity.html. Any violations of this policy may result in failure of the course and being reported to the head of the department. **Not acknowledging collaboration or sources for answers may result in failure of homework or the class along with possible additional review or dismissal by the Department and University.**

Problem 1a

Determine which of the functions are bijective from the reals to the reals. To do this, first prove (or disprove) they are one-to-one, and then prove (or disprove) they are onto.

- a) $f(x) = -3x + 4$
- b) $f(x) = -3x^2 + 7$
- c) $f(x) = \frac{x+1}{x+2}$
- d) $f(x) = x^5 + 1$

Problem 1b

Repeat problem 6a, but determine if the functions are bijective from the integers to the integers.

Problem 2

Prove or disprove the following statements concerning compositions functions for functions f and g

- a) The composition of two one-to-one functions (f and g) is on-to-one.
- b) The composition of two onto functions (f and g) is onto.

Problem 3

A bowl contains 1000 blue balls and 1000 red balls.

- a) How many must be chosen before you have three of the same color?
- b) How many must be chosen until you have 10 of the same color?
- c) How many must be chosen until you have at least 3 blue balls and at least 3 red balls?

Problem 4

If there are nine students in a class, show that at least 5 must be male or at least 5 must be female. Also, show that at least three are male or at least 7 are female.

Problem 5

Prove that at a party with at least two people, that there are two people who know the same **number** of people there (not necessarily the same people - just the same number) given that every person at the party knows at least one person. Also, not that nobody can be his or her own friend. You can solve this with a tricky use of the Pigeonhole Principle.

Problem 6

For each of the following sequences of integers, find a simple formula or rule which generates the formula. You can either make it a closed form or recursive rule if possible, or explain the pattern/rule in English.

- a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...
- b) 7, 11, 15, 19, 23, 27, 31, 35, 29, 43, ...
- c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...
- d) 1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, ...
- e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...
- f) 1, 3, 15, 105, 945, 10395, 135135, 207025, 34459425, ...
- g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, ...
- h) 2, 4, 16, 256, 65536, 4294967296, ...

Problem 7

a) Show that for any sequence $a_0, a_1, a_2, \dots, a_n$ of real numbers that $\sum_{j=1}^n (a_j - a_{j-1}) = a_n - a_0$.

b) Use the formula above and the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ to compute $\sum_{j=1}^n \frac{1}{j(j+1)}$.

Problem 8 Use Mathematical Induction to prove the following results:

- a) $\sum_{k=1}^n j(j+1)(j+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers n .
- b) Any postage amount greater than 7 cents can be formed using a combination of 3-cent and 5-cent stamps.
- c) For all integers $n \geq 0$, $6|(n^3 - n)$.
- d) For all integers $n \geq 4$, $n^2 \leq n!$.
- e) A three dimensional chessboard of size $2^n \times 2^n \times 2^n$ with one $1 \times 1 \times 1$ cube missing can be completely covered by $2 \times 2 \times 2$ cubes, each with one $1 \times 1 \times 1$ cube missing.

Problem 9

What is wrong with the following induction proof in which we are trying to prove that for all positive integers n , $\sum_{i=1}^n i = \frac{(n+\frac{1}{2})^2}{2}$?

- i) Basis Step: The formula is true for $n = 1$.
- ii) Assume $\sum_{i=1}^k i = \frac{(k+\frac{1}{2})^2}{2}$.
- iii) Inductive step: From the assumption, we get $\sum_{i=1}^{k+1} i = (k+1) + \frac{(k+\frac{1}{2})^2}{2} = \frac{k^2+k+\frac{1}{4}}{2} + (k+1) = \frac{k^2+3k+\frac{9}{4}}{2} = \frac{(k+\frac{3}{2})^2}{2} = \frac{((k+1)+\frac{1}{2})^2}{2}$. Hence, the statement is true for all positive integers n .

Problem 10 A *complete binary tree* is one where there is one node as the root, and each node has exactly 2 "children". A leaf is a node with no children, and the height refers to how deep the tree goes. So, if there is only one node (the root), the tree has height 0. If there is the root and two children, then there are 2 leaves, and the height is 1. If there are 3 levels, the height is 3, and there is 1 node at height 0, 2 at height 1 and 4 at height 2, etc. Prove the following via induction:

- a) A complete binary tree of height h has 2^h leaves (or there are 2^h nodes at height h).
- b) A complete binary tree of height h has $2^{(h+1)} - 1$ nodes. Alternatively, you can prove that $1 + 2 + 4 + \dots + 2^h = 2^{(h+1)} - 1$.

Problem 11 There are six different airlines that fly from New York to Denver and seven that fly from Denver to San Antonio. How many different possibilities are there for a trip from New York to San Antonio via Denver, when an airline is picked for the flight to Denver and an airline is picked for the continuing flight to San Antonio?

Problem 12 How many license plates can be made considering that there are 50 different states in the U.S., assuming that each plate can only be made using either three letters followed by three digits or four letters followed by two

digits?

Problem 13 Every student in a Discrete Math class is either a computer science major or a mathematics major or is a double major in Math/C.S.. If there are 38 C.S. majors, 23 Math majors, and 7 double majors in the class, how many students are there total?

BONUS Problems

A) You go to a party to take a break from all of your Discrete Math homework and drink away your math blues. Unfortunately, at the party you cannot shake your desire to do some problem-solving (and who can blame you when it's so fun!). You notice that everyone at the party has shaken hands with three other people except for you - you've shaken the hands of only one other person (the host). You start to wonder:

What is the smallest number of people that could be at the party?

Could 21 people be at a party such as this?

Is there a pattern for how many people could be at this party?

Is doing math problems resulting in nobody wanting to shake your hand?

B) You are told that you can climb a ladder one step at a time or two steps at a time, so in order to get to step 2, there are two ways you can proceed: you can do one big step to step 2 or two small steps of size one. Similarly, there are three ways to get to step 3 (what are they). How many ways can you get to step 100? How about step n ?

C) How many ways can you spell "WAS IT A CAT I SAW" from the following:

W
W a W
W a s a W
W a s i s a W
W a s i t i s a W
W a s i t a t i s a W
W a s i t a c a t i s a W
W a s i t a t i s a W
W a s i t i s a W
W a s i s a W
W a s a W
W a W
W