Homework 2 - Discrete Math - Due 7/05/16

Assigned: 06/21/2016
Due: 07/05/16

Please make sure to clearly write your name at the top of your hand-in. Also, indicate if you worked with anybody and also indicate how many hours total you worked on the homework. This looks like more homework than it is since many problems are quite simple and others have solutions in the back. Feel free to discuss any problems (including the bonuses) on the class mailing list. I am also required to remind all students of the academic integrity policy at `http://www.cs.nyu.edu/web/Academic/Graduate/academic_integrity.html`. Any violations of this policy may result in failure of the course and being reported to the head of the department.

Reading

- Read Chapter 4.
- If you have Polya's book, read that as well at a leisurely pace.

Problem 1
Prove the following two propositions using a contradiction approach:

a) For any integer $a \geq 2$ and any integer $k \geq 2$, if k divides a then k does not divide $(a+1)$.
b) the square root of 2 is irrational. Follow the approach in the book and make sure you comprehend it!

Problem 2 To prove a biconditional statement, prove both directions. That is, to prove $p \leftrightarrow q$, you have to prove both cases: $p \rightarrow q$ and $q \rightarrow p$. For the following statement, prove the forward direction using a direct proof and the reverse direction using proof by contraposition.

For any integer n, n^2 is odd if and only if n is odd.

Problem 3 Prove or disprove the following statements involving the floor and ceiling functions.

a) For all real x, $\lfloor \frac{\lfloor \frac{x}{2} \rfloor}{2} \rfloor = \lfloor \frac{x}{4} \rfloor$.

b) For positive integers n and k, $\lceil \frac{n}{k} \rceil = \lfloor \frac{(n-1)}{k} \rfloor + 1$.

Problem 4

If the following statements are true, prove them. If not, disprove via counterexample.

a) If a divides b and a divides c, then a divides (b+c).
b) If a divides (b+c), then a divides b and a divides c.

Problem 5

The following definitions follow from class:

For integers a and b (not equal to zero), the largest integer d that divides both a and b is called the *greatest common divisor* of a and b, and we write this as $gcd(a, b) = d$. For example, $gcd(24, 36) = 12$.

For integers a and b (not equal to zero), the smallest positive integer c that is divisible by both a and b is called the *least common multiple* of a and b, and we write this as $lcm(a, b) = c$. For example, $lcm(24, 36) = 72$.

Once you get a feeling for the above definitions (try some problems from the book), prove or disprove the following statement:

For positive integers a and b, $a \cdot b = gcd(a, b) \cdot lcm(a, b)$.

(Hint: Consider the prime factorizations of a and b. Namely, $a = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} ... p_n^{b_n}$. Think about how the prime factorizations of a and b relate to $gcd(a, b)$ and $lcm(a, b)$.)

Problem 6

Assume you have a 16 by 16 checkerboard and 128 dominoes, where each domino covers two squares perfectly. Convince yourself that you can cover the whole board with the 128 dominoes (we will soon show you can do this by induction, but just convince yourself for now that it's true). Now, if we remove the top left and bottom right corners, can we still cover the whole board with 127 dominoes?

Problem 7

As in class, we discussed **Russell's Paradox**, so this should be simple review, but I want you to be able to put the answer into plain English. Let S

be a set that contains all sets X such that $X \notin X$. That is, $S = \{X | X \notin X\}$.

a) Show why assuming $S \in S$ leads to a contradiction.
b) Show why assuming $S \notin S$ leads to a contradiction.

Problem 8a Prove the following via direct proof. That is, show that each side of the equation is a subset of the other side (two cases) using an arbitrary particular element x as we did in class for DeMorgan's Law for two sets.

$A^c \cup B^c \cup C^c = (A \cap B \cap C)^c$

Problem 8b Show how the identity above in 8a can be proved using two steps of DeMorgan's Law along with some other basic set rules (An algebraic proof).

Problem 9
Draw Venn diagrams for the following combinations of sets A, B, and C.

a) $A \cap (B \cup C)$
b) $A^c \cap B^c \cap C^c$
c) $(A - B) \cup (A - C) \cup (B - C)$

Problem 10
What can you conclude about A and B if the following are true?

a) $A \cup B = A$
b) $A - B = A$
c) $A \cap B = B \cap A$
d) $A - B = B - A$
e) $A \cap B = A$

Problem 11
Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

BONUS Problems

I I arrange the first ten digits as 8, 5, 4, 9, 1, 7, 6, 3, 2, 0. What is the pattern?

II A woman walks into a deli and buys two cans of soda: root beer and gingerale. When she pays for the sodas, the clerk givers her change and a receipt. She then proceeds to draw a small triangle on the bottom of the receipt as well as the following equation: $3*13 = 39$. The clerk notices all l of this, and says to the woman, "I notice that you're a firefighter." How did the clerk know this?

III Here's one that's somewhat unrelated, but it's good for your brain skills: Imagine you have two hourglasses which can tell you when 4 minutes have passed and when 7 minutes have passed. However, you want to know when 9 minutes have passed. Sitting there with your 7 and 4 minute hourglasses, how will you know when 9 minutes have passed? Draw pictures if you need help explaining how you do it.

IV) 8 players want to play chess against each other in order to rank who is the best. It is assumed if player A beats player B and B beats C then A would beat C. Playing one match takes 1 hour and players can play several matches in a row if need be (they are very strong mentally). Overall, you want to rank them from best to worst in 20 hours or less, but you only have one table. Can you do it? Can you do it in less than 18 hours? How about less than 17 hours? The chess gods have come to the rescue and delivered 3 more tables, so now you have 4 tables. Can you rank the players in 6 hours or less? (The answer is yes - come up with the solution). One 6-hour solution has 2 only tables being used during two of the hours. Is there any way to rank the players in 5 hours? (As far as I know, this problem is unsolved, so don't go crazy).