Discrete Mathematics
07-21-2015

Harper Langston

New York University

Sequences

- Sequence is a set of (usually infinite number of) ordered elements: a_1, a_2, …, a_n, …
- Each individual element a_k is called a term, where k is called an index
- Sequences can be computed using an explicit formula: $a_k = k * (k + 1)$ for $k > 1$
- Alternate sign sequences
- Finding an explicit formula given initial terms of the sequence: 1, -1/4, 1/9, -1/16, 1/25, -1/36, …
- Sequence is (most often) represented in a computer program as a single-dimensional array

Sequence Operations

- Summation: Σ, expanded form, limits (lower, upper) of summation, dummy index
- Change of index inside summation
- Product: Π , expanded form, limits (lower, upper) of product, dummy index
- Factorial: n!, n! = n * (n – 1)!

Sequences

- Geometric sequence:
  $a, ar, ar^2, ar^3, \ldots, ar^n$
- Arithmetic sequence:
  $a, a+d, a+2d, \ldots, a+nd$
- Sum of geometric sequence:
  $\sum_{0\text{->}n} ar^k$
- Sum of arithmetic sequence:
  $\sum_{0\text{->}n} a+kd$

Review Mathematical Induction

- Principle of Mathematical Induction:

  Let P(n) be a predicate that is defined for integers n and let a be some integer. If the following two premises are true:

  P(a) is a true

  $\forall k \geq a, P(k) \rightarrow P(k + 1)$

  then the following conclusion is true as well

  P(n) is true for all $n \geq a$

Applications of Mathematical Induction

- Show that $1 + 2 + \ldots + n = n * (n + 1) / 2$ (Prove on board)
- Sum of geometric series:

  $r^0 + r^1 + \ldots + r^n = (r^{n+1} - 1) / (r - 1)$

  (Prove on board)

Examples that Can be Proved with Mathematical Induction

- Show that $2^{2n} - 1$ is divisible by 3 (in book)
- Show (on board) that for $n > 2$: $2n + 1 < 2^n$
- Show that $x^n - y^n$ is divisible by $x - y$
- Show that $n^3 - n$ is divisible by 6 (similar to book problem)

Strong Mathematical Induction

- Utilization of predicates P(a), P(a + 1), …, P(n) to show P(n + 1).
- Variation of normal M.I., but basis may contain several proofs and in assumption, truth assumed for all values through from base to k.
- Examples:
- Any integer greater than 1 is divisible by a prime
- Existence and Uniqueness of binary integer representation (Read in book)

Well-Ordering Principle

- Well-ordering principle for integers: a set of integers that are bounded from below (all elements are greater than a fixed integer) contains a least element
- Example:
- Existence of quotient-remainder representation of an integer n against integer d

Counting and Probability

- Coin tossing
- Random process
- Sample space is the set of all possible outcomes of a random process. An event is a subset of a sample space
- Probability of an event is the ratio between the number of outcomes that satisfy the event to the total number of possible outcomes
  $P(E) = N(E)/N(S)$ for event E and sample space S
- Rolling a pair of dice and card deck as sample random processes

Possibility Trees

- Teams A and B are to play each other repeatedly until one wins two games in a row or a total three games.
  - What is the probability that five games will be needed to determine the winner?
- Suppose there are 4 I/O units and 3 CPUs. In how many ways can I/Os and CPUs be attached to each other when there are no restrictions?

Multiplication Rule

- Multiplication rule: if an operation consists of k steps each of which can be performed in n_i ways (i = 1, 2, …, k), then the entire operation can be performed in $\prod n_i$ ways.
- Number of PINs
- Number of elements in a Cartesian product
- Number of PINs without repetition
- Number of Input/Output tables for a circuit with n input signals
- Number of iterations in nested loops

Multiplication Rule

- Three officers – a president, a treasurer and a secretary are to be chosen from four people: Alice, Bob, Cindy and Dan. Alice cannot be a president, Either Cindy or Dan must be a secretary. How many ways can the officers be chosen?

Permutations

- A permutation of a set of objects is an ordering of these objects
- The number of permutations of a set of n objects is n! (Examples)
- An r-permutation of a set of n elements is an ordered selection of r elements taken from a set of n elements: P(n, r) (Examples)
- $P(n, r) = n! / (n - r)!$
- Show that $P(n, 2) + P(n, 1) = n^2$

Addition Rule

- If a finite set A is a union of k mutually disjoint sets $A_1, A_2, \ldots, A_k$, then $n(A) = \Sigma n(A_i)$
- Number of words of length no more than 3
- Number of 3-digit integers divisible by 5

Difference Rule

- If A is a finite set and B is its subset, then $n(A - B) = n(A) - n(B)$
- How many PINS contain repeated symbols?
- So, $P(A^c) = 1 - P(A)$ (Example for PINS)
- How many students are needed so that the probability of two of them having the same birthday equals 0.5?

Inclusion/Exclusion Rule

- 2 sets
- 3 sets

Combinations

- An r-combination of a set of n elements is a subset of r elements: $C(n, r)$
- Permutation is an ordered selection, combination is an unordered selection
- Quantitative relationship between permutations and combinations: $P(n, r) = C(n, r) * r!$
- Permutations of a set with repeated elements
- Double counting

Team Selection Problems

- There are 12 people, 5 men and 7 women, to work on a project:
  - How many 5-person teams can be chosen?
  - If two people insist on working together (or not working at all), how many 5-person teams can be chosen?
  - If two people insist on not working together, how many 5-person teams can be chosen?
  - How many 5-person teams consist of 3 men and 2 women?
  - How many 5-person teams contain at least 1 man?
  - How many 5-person teams contain at most 1 man?

Poker Problems

- What is a probability to contain one pair?
- What is a probability to contain two pairs?
- What is a probability to contain a triple?
- What is a probability to contain royal flush?
- What is a probability to contain straight flush?
- What is a probability to contain straight?
- What is a probability to contain flush?
- What is a probability to contain full house?

Combinations with Repetition

- An r-combination with repetition allowed is an unordered selection of elements where some elements can be repeated
- The number of r-combinations with repetition allowed from a set of n elements is $C(r + n - 1, r)$
- Soft drink example

Algebra of Combinations and Pascal's Triangle

- The number of r-combinations from a set of n elements equals the number of (n – r)-combinations from the same set.
- Pascal's triangle: $C(n + 1, r) = C(n, r – 1) + C(n, r)$
- $C(n,r) = C(n,n-r)$

Probability Axioms

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - What if A and B mutually disjoint? (Then $P(A \cap B) = 0$)

Conditional Probability

- For events A and B in sample space S if $P(A) \neq 0$, then the probability of B given A is:
  $P(A \mid B) = P(A \cap B)/P(A)$
- Example with Urn and Balls:
  - An urn contains 5 blue and

Conditional Probability Example

- An urn contains 5 blue and 7 gray balls. 2 are chosen at random.
  - What is the probability they are blue?
  - Probability first is not blue but second is?
  - Probability second ball is blue?
  - Probability at least one ball is blue?
  - Probability neither ball is blue?

Conditional Probability Extended

- Imagine one urn contains 3 blue and 4 gray balls and a second urn contains 5 blue and 3 gray balls
- Choose an urn randomly and then choose a ball.
- What is the probability that if the ball is blue that it came from the first urn?

Bayes' Theorem

- Extended version of last example.
- If S, our sample space, is the union of n mutually disjoint events, B1, B2, …, Bn and A is an even in S with $P(A) \neq 0$ and k is an integer between 1 and n, then:

$$P(Bk \mid A) = \frac{P(A \mid Bk) * P(Bk)}{P(A \mid B1)*P(B1) + \ldots + P(A \mid Bn)*P(Bn)}.$$

Application: Medical Tests (false positives, etc.)

Independent Events

- If A and B are independent events, $P(A \cap B) = P(A)*P(B)$
- If C is also independent of A and B $P(A \cap B \cap C) = P(A)*P(B)*P(C)$
- Difference from Conditional Probability can be seen via Russian Roulette example.