Homework 5 - Discrete Math - Due 08/11/15

Assigned: 07/26/2015
Due: 8/11/15

Please make sure to clearly write your name at the top of your hand-in. Also, **indicate if you worked with anybody and also indicate how many hours total you worked on the homework**. Feel free to discuss any problems (including the bonuses) on the class mailing list. I am also required to remind all students of the academic integrity policy at `http://www.cs.nyu.edu/web/Academic/Graduate/academic_integrity.html`. Any violations of this policy may result in failure of the course and being reported to the head of the department. **Not acknowledging collaboration or sources for answers may result in failure of homework or the class along with possible additional review or dismissal by the Department and University**.

Problem 1

For each of the following sequences of integers, find a simple formula or rule which generates the formula. You can either make it a closed form or recursive rule if possible, or explain the pattern/rule in English.

a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...
b) 7, 11, 15, 19, 23, 27, 31, 35, 29, 43, ...
c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...
d) 1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, ...
e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...
f) 1, 3, 15, 105, 945, 10395, 135135, 207025, 34459425, ...
g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, ...
h) 2, 4, 16, 256, 65536, 4294967296, ...

Problem 2

a) Show that for any sequence $a_0, a_1, a_2, ..., a_n$ of real numbers that $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$.

b) Use the formula above and the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ to compute $\sum_{j=1}^{n} \frac{1}{j(j+1)}$.

Problem 3 Use Mathematical Induction to prove the following results:

a) $\sum_{k=1}^{n} j(j+1)(j+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers n.

b) Any postage amount greater than 7 cents can be formed using a combinatino of 3-cent and 5-cent stamps.

c) For all integers $n \geq 0$, $6|(n^3 - n)$.

d) For all integers $n \geq 4$, $n^2 \leq n!$.

e) A three dimensional chessboard of size 2^n x 2^n x 2^n with one 1 x 1 x 1 cube missing can be completely covered by 2 x 2 x 2 cubes, each with one 1 x 1 x 1 cube missing.

Problem 4

What is wrong with the following induction proof in which we are trying to prove that for all positive integers n, $\sum_{i=1}^{n} i = \frac{(n+\frac{1}{2})^2}{2}$?

i) Basis Step: The formula is true for $n = 1$.

ii) Assume $\sum_{i=1}^{k} i = \frac{(k+\frac{1}{2})^2}{2}$.

iii) Inductive step: From the assumption, we get $\sum_{i=1}^{k+1} i = (k+1) + (\frac{(k+\frac{1}{2})^2}{2}) = \frac{k^2+k+\frac{1}{4}}{2} + (k+1) = \frac{k^2+3k+\frac{9}{4}}{2} = \frac{(k+\frac{3}{2})^2}{2} = \frac{((k+1)+\frac{1}{2})^2}{2}$. Hence, the statement is true for all positive inetegers n.

Problem 5 A *complete binary tree* is one where there is one node as the root, and each node has exactly 2 ”children”. A leaf is a node with no children, and the height refers to how deep the tree goes. So, if there is only one node (the root), the tree has height 0. If there is the root and two children, then there are 2 leaves, and the height is 1. If there are 3 levels, the height is 3, and there is 1 node at height 0, 2 at height 1 and 4 at height 2, etc. Prove the following via induction:

a) A complete binary tree of height h has 2^h leaves (or there are 2^h nodes at height h). b) A complete binary tree of height h has $2^{(h+1)} - 1$ nodes. Alternatively, you can prove that $1 + 2 + 4 + ... + 2^h = 2^{(h+1)} - 1$.

Problem 6 There are six different airlines that fly from New York to Denver and seven that fly from Denver to San Antonio. How many different possibilities are there for a trip from New York to San Antonio via Denver, when an airline is picked for the flight to Denver and an airline is picked for the continuing flight to San Antonio?

Problem 7 How many license plates can be made considering that there are 50 different states in the U.S., assuming that each plate can only be made using either three letters followed by three digits or four letters followed by two digits?

Problem 8 Every student in a Discrete Math class is either a computer science major or a mathematics major or is a double major in Math/C.S.. If there are 38 C.S. majors, 23 Math majors, and 7 double majors in the class, how many students are there total?

Problem 9 A professor writes 40 true/false questions for a test, and 17 of them are true. If the questions can be positioned in any order, how many different answer keys are possible?

Problem 10 Seven women and nine men are on the faculty in the Mathematics department at a school. How many ways are there to select a committee of five members of the department if at least one must be a woman and at least one must be a man?

Problem 11 How many strings of length 10 consisting of lower-case letters from the English alphabet contain the letters **a** and **b**, where **a** is somewhere to the left of **b** in the string, with all the letters distinct?

Problem 12 Find the coefficient of x^5y^8 in the expansion of $(x+y)^{13}$.

Problem 13 Prove the Binomial Theorem using Mathematical Induction. This should be easy to locate online or in a book, so make sure you understand the proof, and cite your source if you found the proof elsewhere.

Problem 14 A croissant shop has plain, cherry, chocolate, almond, apple, cheese, and broccoli croissants. How many ways are there to choose two dozen croissants with at least one plain, at least two cherry, at least three chocolate, at least one almond, at least two apple, and no more than three broccoli croissants?

Problem 15 How many ways are there to distribute 12 identical balls into six bins, numbered one through six (the balls are indistinguishable, but the bins are distinguishable)?

Problem 16 How many ways are there to distribute five distinguishable balls into three indistinguishable bins?

Problem 17 Which is more likely probability-wise: rolling a total of 8 when two dice are rolled, or rolling a total of 8 when three dice are rolled?

Problem 18 From a deck of randomly well-shuffled 52 cards, what is the probability of getting a full-house (three of a kind plus two of a kind)?

Problem 19 If we randomly select a permutation of the 26 lowercase letters of the English alphabet, what is the probability that the first 13 letters of the permutation are in alphabetical order? What is the probability that **z** precedes both **a** and **b**?

Problem 20 What is the conditional probability that a family with two children has two boys. given they have at least one boy? Assume that each of the four possibilities (BB, BG, GB and GG) are equally likely).

Problem 21 Find the probability that a family with five children does not have a boy, if the sexes of the children are independent of one another, and if:

- A boy and a girl are equally likely;
- The probability of having a boy is 0.51;
- The probability that the i^{th} child is a boy is $0.51 - \frac{i}{100}$;

BONUS Problems

A) A woman has two friends who live in different parts of the city from her (one is uptown and one is downtown), but they both live off of the same subway line that runs every 10 minutes in both directions, and they are always on time. She decides which friend to visit based on which train comes first, but no matter what times she shows up, 90 percent of the time, she ends up going downtown. Why?

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            w
          w a w
        w a s a w
      w a s i s a w
    w a s i t i s a w
  w a s i t a t i s a w
w a s i t a c a t i s a w
  w a s i t a t i s a w
    w a s i t i s a w
      w a s i s a w
        w a s a w
          w a w
            w
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B) You are told that you can climb a ladder one step at a time or two steps at a time, so in order to get to step 2, there are two ways you can proceed: you can do one big step to step 2 or two small steps of size one. Similarly, there are three ways to get to step 3 (what are they). How many ways can you get to stepp 100? How about step n?

C) How many ways can you spell "WAS IT A CAT I SAW" from the following:

D) Imagine that you are taking classes in some city at 1st Avenue and 1st Street, but you live at 10th Avenue and 57th Street. You want to walk home by always going west and north. The avenues run north/south and 1st Avenue is the eastern most avenue, then 2nd ave, 3rd ave, 4th ave, 5th ave, 6th ave, 7th ave, 8th ave, 9th ave, 10th ave (10th being the western-most). The streets run east/west, with 1st Street being the sourthermost street and 57th Street being the northernmost street. This city is New York-like, but I am simplifying it to remove Madison, Park, Broadway, etc. So, a map of the area described is a perfect grid. How many ways can you walk home? Given that we covered this at the beginning of class, you just need to find the correct corresponding row in Pascal's Triangle.

E) In a laser fight, you shoot with 33% accuracy, player B shoots with 50% accuracy, and player C shoots with 100% accuracy. The player with the worst accuracy (you) gets to shoot first (there's nothing to hide behind here), followed by player B and then C. Once a player is "shot", they are out. The game continues in this fashion each round (worst percentage shooter shoots first, etc.) until one player is left standing. So for example, if you shoot at player C and miss (2/3 probability this happens), player B shoots at you and hits (1/2 probability) and the player C aims at B and hits (1/1 probability), the probability of this outcome would be $(2/3)(1/2)(1/1) = (1/3)$. So there is a (1/3) chance of this outcome. What is the best strategy for you to possibly become the winner?