HOMEWORK 2 - DISCRETE MATH - DUE 6/30/15

Assigned: 06/15/2015
Due: 06/30/15

Please make sure to clearly write your name at the top of your hand-in. Also, indicate if you worked with anybody and also indicate how many hours total you worked on the homework. This looks like more homework than it is since many problems are quite simple and others have solutions in the back. Feel free to discuss any problems (including the bonuses) on the class mailing list. I am also required to remind all students of the academic integrity policy at `http://www.cs.nyu.edu/web/Academic/Graduate/academic_integrity.html`. Any violations of this policy may result in failure of the course and being reported to the head of the department.

Reading

- Read Chapter 4.
- If you have Polya's book, read that as well at a leisurely pace.

Problem 1 If we were trying to show that $\forall x(P(x) \vee Q(x))$ is equivalent to $\forall xP(X)\forall xQ(X)$, is the following proof valid (Hint: No since they are not equivalent). Find the error or errors

1. $\forall x(P(x) \vee Q(x))$ (Premise)
2. $P(c) \vee Q(c)$ for some value c (Universal Instantiation (1))
3. $P(c)$ (Simplification from (2))
4. $\forall xP(x)$ (Universal Generalization from (3))
5. $Q(c)$ (Simplification from (2))
6. $\forall xQ(x)$ (Universal Generalization from (5))
7. $\forall x(P(x) \vee \forall xQ(x))$ (Conjunction from (4) and (6))

Problem 2 Prove the following statements via direct proof (using the construction methods we covered in class - you can use the sum of two even numbers being even as an example guide).

a) The negation of an odd integer is an odd integer.
b) The product of two odd integers is odd.

Problem 3
Prove the following two propositions using a contradiction approach:

a) For any integer a and any integer k, if k divides a then k does not divide $(a+1)$.

b) the square root of 2 is irrational. Follow the approach in the book and make sur eyou comprehend it!

Problem 4 To prove a biconditional statement, prove both directions. That is, to prove $p \leftrightarrow q$, you have to prove both cases: $p \rightarrow q$ and $q \rightarrow p$. For the following statement, prove the forward direction using a direct proof and the reverse direction using proof by contraposition.

For any integer n, n^2 is odd if and only if n is odd.

Problem 5 Prove or disprove the following statements involving the floor and ceiling functions.

a) For all real x, $\lfloor \frac{\lfloor \frac{x}{2} \rfloor}{2} \rfloor = \lfloor \frac{x}{4} \rfloor$.

b) For positive integers n and k, $\lceil \frac{n}{k} \rceil = \lfloor \frac{(n-1)}{k} \rfloor + 1$.

Problem 6

If the following statements are true, prove them. If not, disprove via counterexample.

a) If a divides b and a divides c, then a divides (b+c).

b) If a divides (b+c), then a divides b and a divides c.

Problem 7

The following definitions follow from class:

For integers a and b (not equal to zero), the largest integer d that divides both a and b is called the *greatest common divisor* of a and b, and we write this as $gcd(a,b) = d$. For example, $gcd(24,36) = 12$.

For integers a and b (not equal to zero), the smallest positive integer c that is divisible by both a and b is called the *least common multiple* of a and b, and we write this as $lcm(a,b) = c$. For example, $lcm(24,36) = 72$.

Once you get a feeling for the above definitions (try some problems from the book), prove or disprove the following statement:

For positive integers a and b, $a \cdot b = gcd(a,b) \cdot lcm(a,b)$.

(Hint: Consider the prime factorizations of a and b. Namely, $a = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} ... p_n^{b_n}$. Think about how the prime factorizations of a and b relate to $gcd(a, b)$ and $lcm(a, b)$.)

Problem 8

Assume you have a 16 by 16 checkerboard and 128 dominos, where each domino covers two squares perfectly. Convince yourself that you can cover the whole board with the 128 dominos (we will soon show you can do this by induction, but just convince yourself for now that it's true). Now, if we remove the top left and bottom right corners, can we still cover the whole board with 127 dominos?

BONUS Problems

I I arrange the first ten digits as 8, 5, 4, 9, 1, 7, 6, 3, 2, 0. What is the pattern?

II A woman walks into a deli and buys two cans of soda: root beer and gingerale. When she pays for the sodas, the clerk givers her change and a receipt. She then proceeds to draw a small triangle on the bottom of the receipt as well as the following equation: 3*13 = 39. The clerk notices al l of this, and says to the woman, "I notice that you're a firefighter." How did the clerk know this?

III Here's one that's somewhat unrelated, but it's good for your brain skills: Imagine you have two hourglasses which can tell you when 4 minutes have passed and when 7 minutes have passed. However, you want to know when 9 minutes have passed. Sitting there with your 7 and 4 minute hourglasses, how will you know when 9 minutes have passed? Draw pictures if you need help explaining how you do it.

IV A cat traps 99 mice in a room and oers them a proposition: They can all be eaten right away or try to save each other by the following. The 99 mice will be lined up so that they are all facing the same direction in a straight line. (So, mouse 99 can see the heads of the other 98 mice, and mouse 1 can see nobody). The cat then says that he will place either a blue cap or a red cap on the top of each mouse's head. After they are lined up, they will have to yell out either red or blue, starting from the back of the line (99), moving forward to the front (1); all mice will be able to hear what everyone behind them yells but nothing else (no tricks with intonation are allowed either). If the color they announce matches that on their head, they will live; otherwise, they are eaten and the others won't know (the cat eats quietly). Again, they will be able to see all of the hats in front of them but not behind them or their own hat color. The mice get together and realize it's better if some live rather than all die. At first they think they could all yell the same color (say blue) and save half of themselves, but there is no guarantee that the colors are split evenly (they might all be red or all red but one or two!). They then realize that they are all pretty good at math, and when one mouse yells a color, they will all be able to hear what he yells. In the end, they decide upon a plan that will save most of them (and that the cat is stupid). How many mice can be saved (what is the minimum and maximum) and why?

Hint : Try solving the problem for a simpler number (i.e., try solving it for 3 mice first and then see if you can extrapolate from there).

V You want to move into a new apartment, but you don't get paid for a month, so you cannot put down a security deposit. Fortunately, you have a bar of gold that is 31 cm long (it is very thin, or your apartment is very expensive) and worth one month's rent, so you make a deal with the landlord. You will give her 1cm of your gold everyday as a deposit for each day. However, every time you cut the bar, it costs you 5 bucks, so you want to cut it as little as possible. Your landlady suggests that you give her 1cm on the first day, 1cm on the second day, and on the third day, you give her one 3cm piece, and she will give you back the two 1cm pieces. On the fourth day you won't have to do any cuts since you will have the two 1cm pieces. But, by the sixth day, you would have to cut again. Obviously, you want to minimize the number of cuts to save money, but you want to make sure you add 1cm to your landlord's pile everyday. Assuming she holds onto all of the gold pieces you give her (so you can trade back and forth), what is the smallest number of cuts you will have to make for the 31 days?