Fundamental Algorithms (Summer 2008)
Partial Solution of Assignment 1

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Disclaimer : There might be typo(s) or even error(s) in the following. If you spot one, please email me (guria@cs.nyu.edu) immediately. Some of the problems might have multiple solutions and only a subset of them will be presented here.

- **CLRS Problems 3-1: Asymptotic behavior of polynomials**

  (a) $[k \geq d]$

$$\begin{aligned} p(n) &= \sum_{i=0}^{i=d} \left(a_i n^i\right) \\ &\leq \sum_{i=0}^{i=d} \left(|a_i| n^i\right), \forall n \geq 0 \\ &\leq \sum_{i=0}^{i=d} \left(|a_i| n^k\right), \forall n \geq 1, \because k \geq d \\ &= \left(\sum_{i=0}^{i=d} |a_i|\right) n^k \end{aligned}$$

  Choose (in the definition of O notation)

$$\begin{aligned} n_o &= 1 \\ c &= \left(\sum_{i=0}^{i=d} |a_i|\right) > 0 \end{aligned}$$

  Now,

$$p(n) \leq cn^k, \forall n \geq n_0$$

  (d) $[k > d]$

  Take any $c > 0$. We will show that there is a choice of n_o (depending on c only) such that

$$p(n) = \sum_{i=0}^{i=d} (a_i n^i) \leq cn^k, \forall n \geq n_o$$

  Choose

$$n_o = max \left\{ 1, \left(\frac{\sum_{i=0}^{i=d} |a_i|}{c} \right)^{\frac{1}{k-d}} \right\}$$

Now,

$$\begin{aligned}
p(n) &= \sum_{i=0}^{i=d}(a_i n^i) \\
&\leq \left(\sum_{i=0}^{i=d}|a_i|\right) n^d, \forall n \geq 1 \\
&\leq \left(\sum_{i=0}^{i=d}|a_i|\right) n^{d-k} n^k \\
&\leq cn^k, \forall n \geq \left(\frac{\sum_{i=0}^{i=d}|a_i|}{c}\right)^{\frac{1}{k-d}}
\end{aligned}$$

(b) $[k \leq d]$

If we consider the polynomial $\sum_{i=0}^{i=d-1}(-a_i n^i)$ as we did in part (b) and use $a_d/2$ in place of c in part (b) and use d in place of k in part (b).

$$n \geq max\left\{1, \frac{\sum_{i=0}^{i=d-1}|a_i|}{a_d/2}\right\}$$

$$\begin{aligned}
&\Rightarrow \sum_{i=0}^{i=d-1}(-a_i n^i) \leq (a_d/2)n^d \because a_d/2 > 0 \\
&\Rightarrow 0 \leq \sum_{i=0}^{i=d-1}(a_i n^i) + (a_d/2)n^d \\
&\Rightarrow (a_d/2)n^d \leq \sum_{i=0}^{i=d}(a_i n^i) \\
&\Rightarrow (a_d/2)n^k \leq \sum_{i=0}^{i=d}(a_i n^i) \because k \leq d, n \geq 1
\end{aligned}$$

Choose

$$\begin{aligned}
n_o &= max\left\{1, \frac{\sum_{i=0}^{i=d-1}|a_i|}{a_d/2}\right\} \\
c &= (a_d/2) > 0
\end{aligned}$$

Now,

$$cn^k \leq p(n), \forall n \geq n_0$$

(c) $[k = d]$

We use results from part (a) and (b). Choose

$$\begin{aligned}
c_1 &= (a_d/2) > 0 \\
c_2 &= \left(\sum_{i=0}^{i=d} |a_i|\right) > 0 \\
n_0 &= max\left\{1, \frac{\sum_{i=0}^{i=d-1} |a_i|}{a_d/2}\right\}
\end{aligned}$$

Now,

$$c_1 n^d \leq p(n) \leq c_2 n^d, \forall n \geq n_0$$

(e) $[k < d]$

Take any $c > 0$. We have to show that there is a choice of n_o (depending on c) such that

$$cn^k \leq p(n), \forall n \geq n_o$$

By Part(b)

$$\begin{aligned}
\sum_{i=0}^{i=d} (a_i n^i) &\geq (a_d/2) n^d, \forall n \geq max\left\{1, \frac{\sum_{i=0}^{i=d-1} |a_i|}{a_d/2}\right\} \\
&\geq cn^k \frac{a_d n^{d-k}}{2c} \\
&\geq cn^k, \forall n \geq \left(\frac{2c}{a_d}\right)^{\frac{1}{d-k}}
\end{aligned}$$

Choose

$$n_0 = max\left\{1, \frac{\sum_{i=0}^{i=d-1} |a_i|}{a_d/2}, \left(\frac{2c}{a_d}\right)^{\frac{1}{d-k}}\right\}$$