Discrete Mathematics
Lecture 2
Logic of Quantified Statements

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Predicates

- A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables
- The domain of a predicate variable is a set of all values that may be substituted in place of the variable
- $P(x)$: x is a student at NYU

Predicates

- If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is the set of all elements in D that make $P(x)$ true when substituted for x. The truth set is denoted as:

  $\{x \in D \mid P(x)\}$

- Let $P(x)$ and $Q(x)$ be predicates with the common domain D. $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$. $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets

Universal Quantifier

- Let P(x) be a predicate with domain D. A universal statement is a statement in the form "$\forall x \in D, P(x)$". It is true iff P(x) is true for every x from D. It is false iff P(x) is false for at least one x from D. A value of x form which P(x) is false is called a counterexample to the universal statement
- Examples
  - D = {1, 2, 3, 4, 5}: $\forall x \in D, x^2 >= x$
  - $\forall x \in R, x^2 >= x$
- Method of exhaustion

Existential Quantifier

- Let P(x) be a predicate with domain D. An existential statement is a statement in the form "$\exists x \in D, P(x)$". It is true iff P(x) is true for at least one x from D. It is false iff P(x) is false for every x from D.
- Examples:
  - $\exists m \in Z, m^2 = m$
  - $E = \{5, 6, 7, 8, 9\}, \exists x \in E, m^2 = m$

Universal Conditional Statement

- Universal conditional statement "∀x, if P(x) then Q(x)":
  - $\forall x$ R, if $x > 2$, then $x^2 > 4$
- Writing Conditional Statements Formally
- Universal conditional statement is called vacuously true or true by default iff P(x) is false for every x in D

Negation of Quantified Statements

- The negation of a universally quantified statement $\forall x \in D, P(x)$ is $\exists x \in D, \sim P(x)$
- "All balls in the bowl are red" – Vacuosly True Example for Universal Statements
- The negation of an existentially quantified statement $\exists x \in D, P(x)$ is $\forall x \in D, \sim P(x)$
- The negation of a universal conditional statement $\forall x \in D, P(x) \rightarrow Q(x)$ is $\exists x \in D, P(x) \wedge \sim Q(x)$

Exercises

- Write negations for each of the following statements:
  - All dinosaurs are extinct
  - No irrational numbers are integers
  - Some exercises have answers
  - All COBOL programs have at least 20 lines
  - The sum of any two even integers is even
  - The square of any even integer is even
- Let P(x) be some predicate defined for all real numbers x, let:

  $r = \forall x \in Z, P(x)$; $s = \forall x \in Q, P(x)$; $t = \forall x \in R, P(x)$
  - Find P(x) (but not $x \in Z$) so that r is true, but s and t are false
  - Find P(x) so that both r and s are true, but t is false

Variants of Conditionals

- Contrapositive
- Converse
- Inverse
- Generalization of relationships from before
- Examples

Necessary and Sufficient Conditions, Only If

- $\forall x$, r(x) is a sufficient condition for s(x) means: $\forall x$, if r(x) then s(x)
- $\forall x$, r(x) is a necessary condition for s(x) means: $\forall x$, if s(x) then r(x)
- $\forall x$, r(x) only if s(x) means: $\forall x$, if r(x) then s(x)

Multiply Quantified Statements

- For all positive numbers x, there exists number y such that y < x
- There exists number x such that for all positive numbers y, y < x
- For all people x there exists person y such that x loves y
- There exists person x such that for all people y, x loves y
- Definition of mathematical limit
- Order of quantifiers matters in some (most) cases

Negation of Multiply Quantified Statements

- The negation of $\forall x, \exists y, P(x, y)$

  is logically equivalent to $\exists x, \forall y, \sim P(x, y)$
- The negation of $\exists x, \forall y, P(x, y)$

  is logically equivalent to $\forall x, \exists y, \sim P(x, y)$

Prolog Programming Language

- Can use parts of logic as programming lang.
- Simple statements:

  isabove(g, b), color(g, gray)
- Quantified statements:

  if isabove(X, Y) and isabove(Y, Z) then isabove(X, Z)
- Questions:

  ?color(b, blue), ?isabove(X, w)

Exercises

- Rewrite $\exists! x \in D, P(x)$ without using the symbol $\exists!$
- Determine whether a pair of quantified statements have the same truth values
  - $\forall x \in D, (P(x) \wedge Q(x))$ vs $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$
  - $\exists x \in D, (P(x) \wedge Q(x))$ vs $(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$
  - $\forall x \in D, (P(x) \vee Q(x))$ vs $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$
  - $\exists x \in D, (P(x) \vee Q(x))$ vs $(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

Arguments with Quantified Statements

- Rule of universal instantiation: if some property is true of everything in the domain, then this property is true for any subset in the domain
- Universal Modus Ponens:
  - Premises: ($\forall$x, if P(x) then Q(x)); P(a) for some a
  - Conclusion: Q(a)
- Universal Modus Tollens:
  - Premises: ($\forall$x, if P(x) then Q(x)); ~Q(a) for some a
  - Conclusion: ~P(a)
- Converse and inverse errors

Validity of Arguments using Diagrams

- Premises: All human beings are mortal; Zeus is not mortal. Conclusion: Zeus is not a human being
- Premises: All human beings are mortal; Felix is mortal. Conclusion: Felix is a human being
- Premises: No polynomial functions have horizontal asymptotes; This function has a horizontal asymptote. Conclusion: This function is not a polynomial