Chapter 1

Mathematical Background

Broadly speaking, this text concerns what can and cannot be computed, and when something can be computed how simply and efficiently it can be done. As you will see later in this chapter, there are precisely defined problems which cannot be solved computationally, regardless of how fast or large a computer is used or how much time is spent on the computation. Likewise, as you will see in a later chapter, there are problems that are effectively infeasible although solvable in principle, for the resources needed would be prohibitive.

To formulate these issues precisely entails appropriate mathematical specifications. Accordingly, this chapter begins by reviewing notation and terminology that will be used throughout the book. This is followed by a quick overview of several proof techniques.

1.1 Definitions and Terminology

1.1.1 Sets

A set is a collection of items. To specify a set, the items are written as a list enclosed by braces, e.g. \( A = \{1, 2, 4\} \). This means that \( A \) is the set containing the items 1, 2, and 4. There is no notion of order in the set listing, so \( \{1, 2, 4\} = \{4, 1, 2\} = \{2, 4, 1\} \), etc. Also, there is no notion of duplicate items, so \( \{1, 2, 1\} = \{1, 2\} \) for example.

The empty set is the set containing no items; it is written as \( \{\} \) or \( \varnothing \) for short.

Notation

- \( x \in A \) means that \( x \) is one of the items contained in set \( A \); \( x \) is called an element of \( A \).

- \( A \cap B \), the intersection of \( A \) and \( B \), denotes the set containing those items that are in both \( A \) and \( B \).

- \( A \cup B \), the union of \( A \) and \( B \), denotes the set containing those items that are in at least one of \( A \) or \( B \).

- \( A \subseteq B \) means that every item in \( A \) is also in \( B \); \( A \) is called a subset of \( B \). When, in addition, there is an item in \( B \) which is not in \( A \), we write \( A \subset B \), or for emphasis \( A \subsetneq B \). \( A \) is then called a strict subset of \( B \).
• $\overline{A}$ denotes the set containing those items not in $A$. For this definition to be meaningful we need a notion of the universe of items at hand, the set $U$ of items, where $A \subseteq U$. Then $\overline{A}$ is the set of items in $U$ but not in $A$.

• $|A|$ denotes the number of items in $A$.

• $A - B$ is the set consisting of items contained in $A$ that are not contained in $B$. Note that $A - B = A \cap \overline{B}$.

**Question 1.** *What is the size of the empty set, $|\phi|$?*

![Venn Diagrams](image)

Figure 1.1: Venn Diagrams. (a) Sets $A$ and $B$ in a universal set $U$. (b) Sets $A$ and $\overline{A}$. (c) Sets $A$ and $B$ (represented by circles) and subsets $A - B$, $A \cap B$, $B - A$ (represented by the regions they label).

**Venn Diagrams.** These are diagrams that are used to illustrate the intersections of collections of sets. They can be useful in helping us see what is going on. Typically, the *Universal Set*, $U$, if present, is shown as a large rectangle. Other sets appear as circular shapes inside this rectangle.

**Power Set.** The power set of $A$ is the collection (or set) of all subsets of $A$; it is written as $2^A$. e.g. $A = \{a, b\}$. $\phi \subseteq A$, $a \subseteq A$, $b \subseteq A$, $\{a, b\} \subseteq A$. So $2^A = \text{PowerSet}(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

**Question 2.** *Show that if $|A| = n$, then $|\text{PowerSet}(A)| = 2^n$. Note that this provides some justification for the notation $2^A$.***