Exercises - Geometric transformations - Newton method

1 Geometric transformations

Exercise 1 – Composition of affine maps
Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ be two affine maps.
Show that $G \circ F$ is an affine map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Exercise 2 – Homogeneous coordinates
Let $a = \langle 1, 2 \rangle$ and $b = \langle -1, 2 \rangle$ be two points in $\mathbb{R}^2$.
2.1 – Give the homogeneous coordinates of $a$ and $b$.
2.2 – Give the homogeneous coordinates of the vector $b - a$.

Exercise 3 – Extended matrices
Let $a = \langle 1, -1 \rangle$ and $b = \langle 2, 1 \rangle$ be two points in $\mathbb{R}^2$.
3.1 – Give the extended matrix of the translation $T_{b-a}$ of vector $b - a$ in $\mathbb{R}^2$.
3.2 – Give the extended matrix of the rotation $R_{a, \frac{\pi}{3}}$ of angle $\frac{\pi}{3}$ around $a$ in $\mathbb{R}^2$.
3.3 – Give the image of $c = \langle -1, 2 \rangle$ by $R_{a, \frac{\pi}{3}}$.

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Exercise 4 – Intersection of 3 spheres in $\mathbb{R}^3$.
Let $a = \langle a_1, a_2, a_3 \rangle$ be a point in $\mathbb{R}^3$ and $d \geq 0$ be a real number. The implicit representation of the points of the sphere centered in $a$ with radius $d$ is
\[
\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 | (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 - d^2 = 0 \}.
\]
4.1 – Give a function $f_1 : \mathbb{R}^3 \to \mathbb{R}$ involving the coordinates $x_1, x_2, x_3$ of a point in $\mathbb{R}^3$ that vanishes exactly when $x_1, x_2, x_3$ are the coordinates of a point of the sphere centered in $\langle 0, 0, 0 \rangle$ with radius 1.
4.2 – Give a function $f_2 : \mathbb{R}^3 \to \mathbb{R}$ involving the coordinates $x_1, x_2, x_3$ of a point in $\mathbb{R}^3$ that vanishes exactly when $x_1, x_2, x_3$ are the coordinates of a point of the sphere centered in $\langle 2, 0, 0 \rangle$ with radius 2.
4.3 – Give a function $f_3 : \mathbb{R}^3 \to \mathbb{R}$ involving the coordinates $x_1, x_2, x_3$ of a point in $\mathbb{R}^3$ that vanishes exactly when $x_1, x_2, x_3$ are the coordinates of a point of the sphere centered in $\langle 1, -1, 0 \rangle$ with radius $\sqrt{2}$. 
4.4 – Give a function $F : \mathbb{R}^3 \to \mathbb{R}^3$ that vanishes exactly on the intersections of the three spheres defined above.

**Exercise 5** – Partial derivatives and jacobian matrix. Let $f_3 : \mathbb{R}^3 \to \mathbb{R}$ be the function defined in question 4.3 above. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined in question 4.4 above.

5.1 – Give the three partial derivatives $\frac{\partial f_3}{\partial x_1}, \frac{\partial f_3}{\partial x_2}, \frac{\partial f_3}{\partial x_3}$ in function of $x_1, x_2, x_3$.

5.2 – Give the jacobian matrix $J_F(x_1, x_2, x_3)$ of $F$.

5.3 – Give the jacobian matrix of $F$ evaluated in $\langle 1, 1, 1 \rangle$ (i.e. give $J_F(1, 1, 1)$).