Exercises - Affine Geometry

Exercise 1 – Give an implicit representation for the plane passing through \( \mathbf{a} = (1, 2, 3), \mathbf{b} = (-1, 2, -3), \mathbf{c} = (1, 1, 1) \).

Use the cross product to find a basis for the orthogonal complement of \( \text{Span}(\mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}) \): \( \mathbf{v} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (-6, -4, 2) \), and \( \langle \mathbf{v} \rangle \) is a basis for this orthogonal complement.

Let \( A = \begin{bmatrix} -6 & -4 & 2 \end{bmatrix} \). The plane passing through \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) is implicitly defined as \( \{ \mathbf{u} \in \mathbb{R}^3 | \mathbf{A} \mathbf{u} = \mathbf{Aa} \} \).

Exercise 2 – Give an implicit representation for the line passing through \( \mathbf{d} = (1, 0, 0) \) and \( \mathbf{e} = (0, 1, 0) \).

\( \langle (1, 1, 0), (0, 0, 1) \rangle \) is a basis for the orthogonal complement of \( \text{Span}(\langle \mathbf{e} - \mathbf{d} \rangle) \). Let \( B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \).

The line passing through \( \mathbf{d}, \mathbf{e} \) is implicitly defined as \( \{ \mathbf{u} \in \mathbb{R}^3 | B \mathbf{u} = B \mathbf{d} \} \).

Exercise 3 – Give the coordinates of the intersection of the plane passing through \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) with the line passing through \( \mathbf{d}, \mathbf{e} \) where \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \) are defined as above.

It is the solution of the equation

\[
\begin{bmatrix}
-6 & -4 & 2 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}
\end{bmatrix}
= 
\begin{bmatrix}
-8 \\
1 \\
0
\end{bmatrix}
\]

Exercise 4 – Let \( \mathbf{u} \) and \( \mathbf{v} \) be to vectors of \( \mathbb{R}^n \).

4.1 – Show that \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are colinear.

Suppose first that \( \mathbf{u} \) and \( \mathbf{v} \) are colinear: there exist \( t \in \mathbb{R} \) so that \( \mathbf{v} = t \mathbf{u} \).

Then \( \mathbf{u} \times \mathbf{v} = 
\begin{bmatrix}
\mathbf{u}[2] \mathbf{v}[3] - \mathbf{u}[3] \mathbf{v}[2] \\
\mathbf{u}[3] \mathbf{v}[1] - \mathbf{u}[1] \mathbf{v}[3] \\
\mathbf{u}[1] \mathbf{v}[2] - \mathbf{u}[2] \mathbf{v}[1]
\end{bmatrix}
= 
\begin{bmatrix}
t \mathbf{u}[2] \mathbf{u}[3] - t \mathbf{u}[3] \mathbf{u}[2] \\
t \mathbf{u}[3] \mathbf{u}[1] - t \mathbf{u}[1] \mathbf{u}[3] \\
t \mathbf{u}[1] \mathbf{u}[2] - t \mathbf{u}[2] \mathbf{u}[1]
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Suppose now that \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) and all the components of \( \mathbf{v} \) are non-zero. then

\[
\begin{align*}
\mathbf{u}[2] \mathbf{v}[3] - \mathbf{u}[3] \mathbf{v}[2] &= 0 \\
\mathbf{u}[3] \mathbf{v}[1] - \mathbf{u}[1] \mathbf{v}[3] &= 0 \\
\mathbf{u}[1] \mathbf{v}[2] - \mathbf{u}[2] \mathbf{v}[1] &= 0
\end{align*}
\]

and the system writes down:

\[
\begin{align*}
\mathbf{u}[2] &= \frac{\mathbf{u}[3] \mathbf{v}[2]}{\mathbf{v}[3]} \\
\mathbf{u}[3] &= \frac{\mathbf{u}[3] \mathbf{v}[3]}{\mathbf{v}[3]} \\
\mathbf{u}[1] &= \frac{\mathbf{u}[3] \mathbf{v}[1]}{\mathbf{v}[3]}
\end{align*}
\]

Suppose now that \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) and one component of \( \mathbf{v} \) is zero; Suppose \( \mathbf{v}[3] = 0 \). Then one has

\[
\begin{align*}
\mathbf{u}[3] \mathbf{v}[2] &= 0 \\
\mathbf{u}[3] \mathbf{v}[1] &= 0 \\
\mathbf{u}[1] \mathbf{v}[2] - \mathbf{u}[2] \mathbf{v}[1] &= 0
\end{align*}
\]
Since \( \vec{v}[2] \) and \( \vec{v}[1] \) are different from zero, we get \( \vec{u}[3] = 0 \) and the system writes down:
\[ \vec{u}[1]\vec{v}[2] - \vec{u}[2]\vec{v}[1] = 0 \]
which means that \( \vec{u} \) and \( \vec{v} \) are colinear.

Suppose now that \( \vec{u} \times \vec{v} = \vec{0} \) and two components of \( \vec{v} \) are zero; Suppose \( \vec{v}[3] = \vec{v}[2] = 0 \). Then one has
\[
\begin{align*}
\vec{u}[3]\vec{v}[1] &= 0 \\
\vec{u}[2]\vec{v}[1] &= 0
\end{align*}
\]
Since \( \vec{v}[1] \) is different from zero, we get \( \vec{u}[3] = \vec{u}[2] = 0 \) and \( \vec{u} \) and \( \vec{v} \) are colinear.

If the three components of \( \vec{v} \) are zero, one has \( \vec{v} = 0\vec{u} \) and \( \vec{u} \) and \( \vec{v} \) are colinear.

The other cases are symmetric.

4.2 – Show that \( \vec{u} \times \vec{v} \) is orthogonal to both \( \vec{u} \) and \( \vec{v} \).

It suffices to compute \((\vec{u} \times \vec{v}) \cdot \vec{u}\) and \((\vec{u} \times \vec{v}) \cdot \vec{v}\).

Exercise 5 – Implement in Matlab the procedure basisForOrthoComp of the lecture on affine geometry.