Exercises - Matrices

**Exercise 1** – Compute the product of

- \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \),

- \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \) and \( B = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \),

- \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \),

- \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \) and \( B = 0_{3 \times 3} \).

**Exercise 2** – Let \( A \) be an \( m \times n \) matrix and \( B \) be an \( n \times p \) matrix. Show that \((AB)^t = B^tA^t\)

One has, \( \forall 1 \leq i \leq m \) and \( \forall 1 \leq j \leq p \),

\[
((AB)^t)[i,j] = (AB)[j,i] = A[j,:] \cdot B[:,i] = A^t[:,j] \cdot B^t[i,:] = B^t[i,:] \cdot A^t[:,j] \text{ by commutativity of the dot product}
\]

\[
= (B^tA^t)[i,j]
\]

**Exercise 3** – Let \( \theta \in [0, 2\pi[, \text{ and consider the } 2 \times 2 \text{ matrix}

\[
A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}
\]

Consider the geometric transformation \( F_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( \forall \vec{u} \in \mathbb{R}^2, F_\theta(\vec{u}) = A_\theta \vec{u} \).

3.1 – Which geometric transformation transformation is \( F_\theta \)?
a counter-clockwise rotation around \( \vec{0} \) of angle \( \theta \).

3.2 – How can you interpret \( F_\pi \) in terms of geometric transformation?
\( F_\pi \) can be seen either as a counter-clockwise rotation around \( \vec{0} \) of angle \( \theta \), or as a symmetry with respect to \( \vec{0} \).

3.3 – Compute the matrices \( A_\pi \) and \( A_\pi \).

\[
A_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \quad A_\pi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]
4.5 – Give the matrix \( A_{\theta}A_{-\theta} \).

\( F_\theta \) is the rotation around \( \vec{0} \) of angle \( \theta \), \( F_{-\theta} \) is the rotation around \( \vec{0} \) of angle \(-\theta\); hence the composition of these transformations is the identity map: \( \forall \vec{u} \in \mathbb{R}^2, F_\theta \circ F_{-\theta}(\vec{u}) = \vec{u} \). As a consequence, the matrix associated to \( F_\theta \circ F_{-\theta} \), which is \( A_{\theta}A_{-\theta} \), is \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

3.5 – For any \( \theta_1 \in [0, 2\pi[ \) and \( \theta_2 \in [0, 2\pi[ \), give the matrix \( A_{\theta_1}A_{\theta_2} \).

\( F_{\theta_1} \) is the rotation around \( \vec{0} \) of angle \( \theta_1 \), \( F_{\theta_2} \) is the rotation around \( \vec{0} \) of angle \( \theta_2 \); hence the composition of these transformations is the rotation around \( \vec{0} \) of angle \( \theta_1 + \theta_2 \): \( A_{\theta_1}A_{\theta_2} = A_{\theta_1+\theta_2} \).

Exercise 4 – Linear maps. Say which maps among the following ones are linear. Give the matrices associated with the linear maps.

4.1 – \( F : \mathbb{R}^2 \to \mathbb{R} \) defined as \( F(\vec{u}) = \vec{u}[1] + \vec{u}[2] \)

It is linear and associated with the \( 1 \times 2 \) matrix \[
\begin{bmatrix}
1 & 1
\end{bmatrix}.
\]

4.2 – \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) defined as \( F(\vec{u}) = \langle \vec{u}[1] + \vec{u}[2], 3 \ast \vec{u}[2] \rangle \)

It is linear and associated with the \( 2 \times 2 \) matrix \[
\begin{bmatrix}
1 & 1 \\
0 & 3
\end{bmatrix}.
\]

4.3 – \( F : \mathbb{R}^2 \to \mathbb{R}^3 \) defined as \( F(\vec{u}) = \langle \vec{u}[1] + \vec{u}[2], 3 \ast \vec{u}[2], 2 \rangle \)

It is not linear because \( F(\vec{0}) = \langle 0, 0, 2 \rangle \neq \vec{0} \)

4.4 – \( F : \mathbb{R}^2 \to \mathbb{R}^3 \) defined as \( F(\vec{u}) = \langle \vec{u}[1] + \vec{u}[2], 3 \ast \vec{u}[2], 4 \ast \vec{u}[2] \rangle \)

It is linear and associated with the \( 3 \times 2 \) matrix \[
\begin{bmatrix}
1 & 1 \\
0 & 3 \\
0 & 4
\end{bmatrix}.
\]

4.5 – \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as the rotation around the \( x \)-axis of angle \( \frac{\pi}{3} \) composed with a dilation of ratio \(-2\).

It is linear and associated with the \( 3 \times 3 \) matrix \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\
0 & \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3})
\end{bmatrix} \begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{bmatrix}.
\]

4.6 – \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as the rotation around the \( x \)-axis of angle \( \frac{\pi}{3} \) composed with a translation of vector \( \vec{v} = \langle 1, 2, 3 \rangle \).

It is not linear because \( F(\vec{0}) = \vec{v} \neq \vec{0} \)

Exercise 5 – \([1] \)

Give examples of \( 2 \times 2 \) matrices \( A \) and \( B \) with non-zero integer elements satisfying:

5.1 – \( A^2 = -I_2 \)

We search \( A \) with the form \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
where \( a, b, c, d \) are strictly positive integer.

\( A \) must satisfy \( AA = -I^2 \), that is

\[
\begin{bmatrix}
a^2 + bc & ab + bd \\
ca + dc & cb + d^2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]

so we have the system of equations:

\[
\begin{aligned}
a^2 + bc &= -1 \\
ab + bd &= 0 \\
ca + dc &= 0 \\
\end{aligned}
\]

\[
\begin{aligned}
bc + d^2 &= -1
\end{aligned}
\]

\(^1\)From Magraret Wright
The second equation rewrites \( b(a + d) = 0 \); since \( b \neq 0 \), one has \( a = -d \).

One can verify that if \( a > 0, b > 0 \), then \( A = \begin{bmatrix} a & b \\ \frac{-a^2}{b} & -a \end{bmatrix} \) verifies \( A^2 = -I_2 \). It suffices to choose strictly positive integers for \( a \) and \( b \) to have examples of matrices \( A \) satisfying \( A^2 = -I_2 \).

5.2 – \( B^2 = 0_{2 \times 2} \)

We search \( B \) with the form \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) where \( a, b, c, d \) are strictly positive integer.

\( B \) must satisfy \( 0_{2 \times 2} \), that is

\[
\begin{bmatrix}
 a^2 + bc & ab + bd \\
 ca + dc & cb + d^2
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

so we have the system of equations:

\[
\begin{aligned}
 a^2 + bc &= 0 \\
 ab + bd &= 0 \\
 ca + dc &= 0 \\
 cb + d^2 &= 0
\end{aligned}
\]

The second equation rewrites \( b(a + d) = 0 \); since \( b \neq 0 \), one has \( a = -d \).

One can verify that if \( a > 0, b > 0 \), then \( B = \begin{bmatrix} a & b \\ \frac{-a^2}{b} & -a \end{bmatrix} \) verifies \( B^2 = 0_{2 \times 2} \). It suffices to choose strictly positive integers for \( a \) and \( b \) to have examples of matrices \( B \) satisfying \( B^2 = 0_{2 \times 2} \).

Exercise 6 – \(^2\) A permutation matrix is a square matrix such that each row and each column has one entry of 1 and the rest are 0s. For instance, the matrix \( M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \) is a \( 6 \times 6 \) permutation matrix.

6.1 – Let \( V \) be the row vector \( [1 \ 2 \ 3 \ 4 \ 5 \ 6] \). Compute \( VM \) and \( MV^t \).

\( VM = [3 \ 4 \ 1 \ 6 \ 5 \ 2] \) and \( MV^t = \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \\ 4 \end{bmatrix} \)

6.2 – Let \( U \) be the row vector \( [4 \ 5 \ 6 \ 3 \ 2 \ 1] \). Give a permutation matrix \( M_1 \) such that \( VM_1 = U \). Give a permutation matrix \( M_2 \) such that \( UM_2 = V \). What do you notice about these two matrices?

\(^2\)From Ernest Davis, Linear Algebra and Probability for Computer Science Applications, CRC Press, 2012; p76
\[
M_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix},
M_2 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

One has:
\[M_1 = M_2^2\] and
\[M_1 M_2 = M_2 M_1 = I_6.\]

**Exercise 7** – Let
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
\text{ and } B = \begin{bmatrix}
4 & 5 & 6 \\
7 & 8 & 9 \\
1 & 2 & 3 \\
\end{bmatrix}.
\]

Give a $3 \times 3$ permutation matrix $M$ such that $MA = B$.
\[
M = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

**Exercise 8** – Matlab - Transitive closure of a graph.

A directed graph $G$ is a couple $(V,E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ is a set and $E$ is a subset of $V \times V$. ($E$ it is a set of couples $(v_i, v_j)$). The elements of $V$ are called vertices and the elements of $E$ are called edges.

Let $v_i, v_j$ be two vertices.

We say that there is an edge from $v_i$ to $v_j$ in $G$ if the edge $(v_i, v_j)$ is in $V$.

We say that there is a path of length 1 from $v_i$ to $v_j$ in $G$ if the edge $(v_i, v_j)$ is in $V$.

If $l > 1$, we say that there is a path of length $l$ from $v_i$ to $v_j$ in $G$ if there exist $v_k \in V$ such that there is a path of length $l - 1$ from $v_i$ to $v_k$ in $G$ and there is an edge from $v_k$ to $v_j$ in $G$.

We say that there is a path from $v_i$ to $v_j$ in $G$ if there exist $l \geq 1$ such that there is a path of length $l$ from $v_i$ to $v_j$ in $G$.

Let $G = (V, E)$ be a graph. We call transitive closure of $G$ the graph $G^* = (V, E^*)$ where $(v_i, v_j) \in E^*$ if and only if there is a path from $v_i$ to $v_j$ in $G$.

We call adjacency matrix of a graph $G = (V, E)$ the $n \times n$ matrix $G$ such that $G[i, j] = 1$ if $(v_i, v_j) \in V$, $G[i, j] = 0$ otherwise.

Recall: $(G^l)[i, j] \neq 0$ if and only if there is a path of length $l$ from $v_i$ to $v_j$ in $G$.

8.1 – In a file `transitive_closure.m`, implement the function `transitive_closure` with prototype

```matlab
function Gstar = transitive_closure( G )
```

computing the adjacency matrix of the transitive closure of the graph which adjacency matrix is $G$.

8.2 – In a file `test_graph.m`, compute the transitive closures of the graphs which adjacency matrices are:
\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

an example of solution can be found here:

https://cs.nyu.edu/courses/spring19/CSCI-GA.1180-001/Exercises/Material/Exercise_graph.zip