Mathematical Techniques for Computer Science Applications

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webpage:
https://cs.nyu.edu/courses/spring19/CSCI-GA.1180-001/
Course 6 - Affine geometry

I The ray-tracing algorithm
   1 The problem
   2 The ray-tracing algorithm

II Additional material in linear algebra

III Affine subspaces of $\mathbb{R}^n$

IV Systems of coordinates

V Intersect ray patch procedure
The problem: computing a 2D image from a 3D scene

Be given:

- A 3D scene
- A camera at a given position in the scene

Compute: the picture the camera would give
The problem: computing a 2D image from a 3D scene

Be given:

- A 3D scene
- A camera at a given position in the scene
- Model:
  - a list of triangles of the space, called *patches*
  - their associated colors

patches = \{ \langle p_1, p_2, p_3 \rangle, 
            \langle p_1, p_2, p_4 \rangle, 
            \langle p_2, p_3, p_4 \rangle, 
            \langle p_1, p_3, p_4 \rangle \}

colors = \{ cyan, 
magenta, 
yellow, 
blue \}
The problem: computing a 2D image from a 3D scene

Be given:

- A 3D scene
  
  Model:
  
  - a list of triangles of the space, called *patches*
  - their associated colors

- A camera at a given position in the scene
  
  Model: the *pinhole model* (*sténopé* in french)
The pinhole model

- A *global* reference: a point \( o \) and an orthonormal basis \( \langle \vec{x}_o, \vec{y}_o, \vec{z}_o \rangle \)
  we will assume \( o = \langle 0, 0, 0 \rangle \) and \( \langle \vec{x}_o, \vec{y}_o, \vec{z}_o \rangle = S \)
The pinhole model

- A *camera* reference: a point \( \mathbf{c} \) and an orthonormal basis \( \mathcal{C} = \langle \vec{x}_c, \vec{y}_c, \vec{z}_c \rangle \)
  - \( \mathbf{c} \): position of the lens, \( \mathcal{C} \): orientation of the lens
  - the line directed by \( \vec{z}_c \) is called the *optical axis*
The pinhole model

- A focal length $f \in \mathbb{R}, f > 0$. The image plane is the plane normal to $\vec{z}_c$, intersecting the optical axis at $p = c + f\vec{z}_c$. 
The pinhole model

- The width and height $w, h \in \mathbb{R}$ defining the sensor as the rectangle of width and height $w, h$ of the image plane centered in $p$. 

\[ w \quad h \]

\[ \vec{y}_c \quad \vec{z}_c \quad \vec{x}_c \]

\[ \vec{e}^1 \quad \vec{e}^2 \quad \vec{e}^3 \]
The pinhole model

- The *resolution* of the sensor: two positive integers $r_w, r_h$
  - the sensor is a rectangular grid of $r_w \times r_h$ pixels
  - each pixel is a rectangular zone of the sensor with size $(w/r_w) \times (h/r_h)$.
  - We note $\text{pix}(i,j)$ the center of the $i,j$-th pixel.
The pinhole model

- The 2D image is the projection of center $\mathbf{c}$ on the sensor of the 3D scene
The problem: computing a 2D image from a 3D scene

- How to compute the color of the pixels?
The problem: computing a 2D image from a 3D scene

- How to compute the color of the pixels?
The problem: computing a 2D image from a 3D scene

- How to deal with visibility problems?
The problem: computing a 2D image from a 3D scene

- How to deal with visibility problems?
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The ray-tracing algorithm

Let $1 \leq i \leq r_w$ and $1 \leq j \leq r_h$. $\text{pix}(i, j)$ is the center of the $i, j$-th pixel.

The ray passing through the $i, j$-th pixel is the half-line $[c, \text{pix}(i, j))$. 

![Diagram](image-url)
The ray-tracing algorithm

Let $1 \leq i \leq r_w$ and $1 \leq j \leq r_h$. $\text{pix}(i, j)$ is the center of the $i, j$-th pixel.

The ray passing through the $i, j$-th pixel is the half-line $[c, \text{pix}(i, j)]$.

Idea of the algorithm: For each pixel $(i, j)$ of the sensor:

1. find the intersections of $[c, \text{pix}(i, j)]$ with the patches of the scene
2. if no such intersection, give to the pixel $(i, j)$ the color white (or the color of the background of the scene);
3. else find the patch for which the intersection with $[c, \text{pix}(i, j)]$ is the closest to $c$
give to the pixel $(i, j)$ the color of this patch.
The ray-tracing algorithm

Let $1 \leq i \leq r_w$ and $1 \leq j \leq r_h$. $\text{pix}(i, j)$ is the center of the $i, j$-th pixel.

The *ray* passing through the $i, j$-th pixel is the *half-line* $[c, \text{pix}(i, j))$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{ray_tracing_diagram.png}
\caption{Ray-tracing algorithm diagram}
\end{figure}
The ray-tracing algorithm

Let $1 \leq i \leq r_w$ and $1 \leq j \leq r_h$. $\text{pix}(i, j)$ is the center of the $i, j$-th pixel.

The ray passing through the $i, j$-th pixel is the half-line $[c, \text{pix}(i, j)]$. 
Algorithm

Global variables: $C = (c, \langle \vec{x}_c, \vec{y}_c, \vec{z}_c \rangle), f, w, h, r_w, r_h$, the scene

Procedure: IntersectRayPatch((i, j), k)

**input:** (i, j) the index of a pixel

$k$ the index of a patch

**output:** if the ray passing through the (i, j)-th pixel intersects the k-th patch, returns the distance between c and that intersection

otherwise $+\infty$
Algorithm

Global variables: \( C = (c, (\vec{x}_c, \vec{y}_c, \vec{z}_c)), f, w, h, r_w, r_h, \) the scene

Procedure: IntersectRayPatch((i, j), k)

input: (i, j) the index of a pixel
       k the index of a patch

output: if the ray passing through the \((i, j)\)-th pixel intersects the \(k\)-th patch, returns the distance between \(c\) and that intersection
         otherwise \(+\infty\)

Procedure: RayTracing

output: the 2D image of the scene

01. for each pixel \((i, j)\) do
02.     \(col \leftarrow\) background color, \(dist = +\infty\)
03.     for each patch do let \(k\) be the index of the patch
04.         \(temp \leftarrow\) IntersectRayPatch((i, j), k)
05.         if \(temp < dist\) then
06.             \(col \leftarrow\) color\((k)\), \(dist \leftarrow temp\)
07.         give to pixel \((i, j)\) the color \(col\)
08. return
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    1  Change of basis
    2  Orthogonal complements
III  Affine subspaces of $\mathbb{R}^n$
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V   Intersect ray patch procedure
Change of coordinates

Let $\mathcal{U}$ be a subspace for $\mathbb{R}^n$, and $B = \langle \vec{b}_1, \ldots, \vec{b}_m \rangle$ be a basis for $\mathcal{U}$. Let $\vec{u} \in \mathcal{U}$ and $\text{Coords}(\vec{u}, B) = \langle u_1, \ldots, u_m \rangle$.

Recall:

$$\vec{u} = u_1 \vec{b}_1 + \ldots + u_m \vec{b}_m$$

How to find $\text{Coords}(\vec{u}, B)$ when knowing $\vec{u}$:
Change of coordinates

Let $\mathcal{U}$ be a subspace for $\mathbb{R}^n$, and $B = \langle \vec{b}_1, \ldots, \vec{b}_m \rangle$ be a basis for $\mathcal{U}$. Let $\vec{u} \in \mathcal{U}$ and $\text{Coords}(\vec{u}, B) = \langle u_1, \ldots, u_m \rangle$.

Recall:

$$\vec{u} = u_1 \vec{b}_1 + \ldots + u_m \vec{b}_m$$

How to find $\text{Coords}(\vec{u}, B)$ when knowing $\vec{u}$:

Let $B = \begin{bmatrix} \vec{b}_1, \ldots, \vec{b}_m \end{bmatrix}$ ($n \times m$ matrix with $\text{Rank}(B) = m$), and consider the system:

$$B\vec{x} = \vec{u}$$

Since $\text{Rank}(B) = m$, $\text{Dim} (\text{Null}(B)) = 0$ (system of category III) and $\vec{u} \in \text{Im}(B)$,

$$B\vec{x} = \vec{u}$$ has as unique solution $\text{Coords}(\vec{u}, B)$
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Complement and orthogonal complements

Let $U$ and $V$ be subspaces of $\mathbb{R}^n$.

Recall: We call **direct sum** of $U$ and $V$ the set:

$$U \oplus V = \{ \vec{w} \in \mathbb{R}^n \mid \exists \vec{u} \in U, \exists \vec{v} \in V \text{ s.t. } \vec{w} = \vec{u} + \vec{v} \}$$

it is a subspace of $\mathbb{R}^n$
Complement and orthogonal complements

Let $U$ and $V$ be subspaces of $\mathbb{R}^n$.

Recall: We call direct sum of $U$ and $V$ the set:

$$U \oplus V = \{ \vec{w} \in \mathbb{R}^n | \exists \vec{u} \in U, \exists \vec{v} \in V \text{ s.t. } \vec{w} = \vec{u} + \vec{v} \}$$

it is a subspace of $\mathbb{R}^n$

Definition:
$U$ and $V$ are complements in $\mathbb{R}^n$ if $U \oplus V = \mathbb{R}^n$ and $U \cap V = \{ \vec{0} \}$.

$U$ and $V$ are orthogonal complements in $\mathbb{R}^n$ if they are complements in $\mathbb{R}^n$, and

$$\forall \vec{u} \in U, \forall \vec{v} \in V, \vec{u} \cdot \vec{v} = 0$$

i.e. vectors in $U$ are orthogonal to vectors in $V$. 
Complement and orthogonal complements: examples

\[ U = \text{Span}(\langle \vec{v}_1, \vec{v}_2 \rangle) \] and \[ V = \text{Span}(\langle \vec{v}_3 \rangle) \] are complement in \( \mathbb{R}^3 \)

\[ \vec{v}_1 = \langle 1, 2, 0 \rangle \]
\[ \vec{v}_2 = \langle -2, 1, 0 \rangle \]
\[ \vec{v}_3 = \langle 1, 1, 1 \rangle \]
Complement and orthogonal complements: examples

\( \mathcal{U} = \text{Span}(\langle \vec{v}_1, \vec{v}_2 \rangle) \) and \( \mathcal{V} = \text{Span}(\langle \vec{v}_3 \rangle) \) are complement in \( \mathbb{R}^3 \)

\( \mathcal{U} = \text{Span}(\langle \vec{v}_1, \vec{v}_2 \rangle) \) and \( \mathcal{V}' = \text{Span}(\langle \vec{v}_3' \rangle) \) are orthogonal complement in \( \mathbb{R}^3 \)

\[\begin{align*}
\vec{v}_1 &= \langle 1, 2, 0 \rangle \\
\vec{v}_2 &= \langle -2, 1, 0 \rangle \\
\vec{v}_3 &= \langle 1, 1, 1 \rangle \\
\vec{v}_3' &= \langle 0, 0, 1 \rangle
\end{align*}\]
Complement and orthogonal complements

Theorem (1)

Let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$ of dimension $m$. Then

(i) $\mathcal{U}$ has a complement $\mathcal{V}$ in $\mathbb{R}^n$, and $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, $\dim(\mathcal{V}) = n - m$

(ii) $\mathcal{U}$ has an orthogonal complement $\mathcal{V}'$ in $\mathbb{R}^n$, and $\mathcal{V}'$ is a subspace of $\mathbb{R}^n$, $\dim(\mathcal{V}') = n - m$

Notation: We note $\mathcal{U}^\perp$ the orthogonal complement of $\mathcal{U}$.

Practical problem: Be given a basis for $\mathcal{U}$, find a basis for $\mathcal{U}^\perp$. 


Recall: projection on an orthogonal family

Let $\mathcal{B} = \langle \vec{b}_1, \ldots, \vec{b}_m \rangle$ be an orthogonal family.

Projection on $\text{Span}(\mathcal{B})$:

$$
\text{proj}(\vec{u}, \mathcal{B}) = \frac{\vec{u} \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1 + \ldots + \frac{\vec{u} \cdot \vec{b}_m}{\vec{b}_m \cdot \vec{b}_m} \vec{b}_m
$$
Finding a basis for $\mathcal{U}^\perp$

Procedure: basisForOrthoComp($\mathcal{B}, \mathcal{F}$)

**input:** a basis $\mathcal{B} = \langle \vec{b}_1, \ldots, \vec{b}_m \rangle$ for $\mathcal{U}$ subspace of $\mathbb{R}^n$,

a basis $\mathcal{F} = \langle \vec{f}_1, \ldots, \vec{f}_n \rangle$ for $\mathbb{R}^n$

**output:** a basis $\mathcal{C} = \langle \vec{c}_1, \ldots, \vec{c}_{n-m} \rangle$ for $\mathcal{U}^\perp$

01. $\mathcal{B}' \leftarrow \langle \vec{b}_1 \rangle$, $\mathcal{C} \leftarrow \emptyset$
02. for $i$ from 2 to $m$ do
03. $\vec{c}_i \leftarrow \vec{b}_i - \text{proj}(\vec{b}_i, \mathcal{B}')$
04. $\mathcal{B}' \leftarrow \mathcal{B}' \cup \langle \vec{c}_i \rangle$
05. endfor  \hspace{1cm} $\triangleleft \mathcal{B}'$ is an ortho. basis for $\mathcal{U}$, see Gram Schmidt
06. for $i$ from 1 to $n$ do
07. $\vec{v} \leftarrow \vec{f}_i - \text{proj}(\vec{f}_i, \mathcal{B}' \cup \mathcal{C})$
08. if $\vec{v} \neq \vec{0}$ then
09. $\mathcal{C} \leftarrow \mathcal{C} \cup \langle \vec{v} \rangle$
10. endif
11. endfor
12. return $\mathcal{C}$  \hspace{1cm} $\triangleleft \mathcal{C}$ is an orthogonal basis for $\mathcal{U}^\perp$
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Interpretation of elements of $\mathbb{R}^n$:

$\mathbb{R}^n$ can be seen as:

- the subspace $\mathbb{R}^n$ of $\mathbb{R}^n$: the set of vectors of $\mathbb{R}^n$, equipped with addition, scalar multiplication and dot product

- the set of points of $\mathbb{R}^n$: equipped with nothing

**Definition:** We call points, and note it with lowercase boldface letters like $\mathbf{c}$, the elements of the set $\mathbb{R}^n$. The coordinates of points are noted as $n$-tuples: $\mathbf{c} = \langle c_1, \ldots, c_n \rangle$.

$0 = \langle 0, \ldots, 0 \rangle$
Arithmetic on points and vectors

Let $\mathbf{a}, \mathbf{b}$ be two points in $\mathbb{R}^n$ and $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$.

We define:

- $\mathbf{a} - \mathbf{b}$ as the vector $\langle a[1] - b[1], \ldots, a[n] - b[n] \rangle$,
- $\mathbf{a} + \vec{u}$ as the point $\langle a[1] + \vec{u}[1], \ldots, a[n] + \vec{u}[n] \rangle$,
- $\vec{u} + \vec{v}$ as the vector $\langle \vec{u}[1] + \vec{v}[1], \ldots, \vec{u}[n] + \vec{v}[n] \rangle$. 

\[ \mathbf{p} = \mathbf{c} + f\vec{z}_c \]
Affine subspaces of $\mathbb{R}^n$

**Definition:** [Affine subspace of $\mathbb{R}^n$]

Let $W \subseteq \mathbb{R}^n$ (W is a subset of $\mathbb{R}^n$). $W$ is an **affine subspace** of $\mathbb{R}^n$ if there exist:

- a (linear) subspace $U$ of $\mathbb{R}^n$,
- a point $p \in \mathbb{R}^n$,

so that

$$\forall v \in W, \exists \tilde{u} \in U \text{ s.t. } v = p + \tilde{u}$$

$$L = \{c + t\tilde{z}_c | t \in \mathbb{R}\}$$

$$\mathcal{L} = \text{Span}(\langle \tilde{z}_c \rangle)$$
Affine subspaces of $\mathbb{R}^n$

**Definition:** [Affine subspace of $\mathbb{R}^n$]
Let $W \subseteq \mathbb{R}^n$ (W is a *subset* of $\mathbb{R}^n$). $W$ is an *affine subspace* of $\mathbb{R}^n$ if there exist:

- a (linear) subspace $U$ of $\mathbb{R}^n$,
- a *point* $p \in \mathbb{R}^n$,

so that

$$\forall v \in W, \exists \bar{u} \in U \text{ s.t. } v = p + \bar{u}$$

We note $W = p + U$.
We call $\text{Dim}(U)$ the *dimension* of $W$, and we note it $\text{Dim}(W)$.

**Remark:** If $W = p + U$ is an *affine subspace* of $\mathbb{R}^n$:

- $p \in W$,
- for any $v \in W$, $W = v + U$
Examples

- The line $L$ passing through $\mathbf{c}$ and $\text{pix}(i, j)$ is an affine subspace of dim 1
  \[ L = \mathbf{c} + \text{Span}(\langle \text{pix}(i, j) - \mathbf{c} \rangle) \]
Examples

• The line $L$ passing through $\mathbf{c}$ and $\mathbf{pix}(i, j)$ is an affine subspace of dim 1.

$$L = \mathbf{c} + \text{Span}(\langle \mathbf{pix}(i, j) - \mathbf{c} \rangle)$$

• The plane $P$ passing through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is an affine subspace of dim 2.

$$P = \mathbf{p}_1 + \text{Span}(\langle \mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1 \rangle)$$
Examples

- The line $L$ passing through $c$ and $pix(i,j)$ is an affine subspace of dim 1
  \[ L = c + \text{Span}(\langle pix(i,j) - c \rangle) \]

- The plane $P$ passing through $p_1, p_2, p_3$ is an affine subspace of dim 2 if $p_1, p_2, p_3$ are not in the same line
  \[ P = p_1 + \text{Span}(\langle p_2 - p_1, p_3 - p_1 \rangle) \]
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Affine subspaces are sols of systems of linear equations

Notation: A \( p \times q \) matrix, \( \mathbf{v} \in \mathbb{R}^q \), \( A\mathbf{v} \) is the \( p \)-dim vectors \( A\langle \mathbf{v}[1], \ldots, \mathbf{v}[q] \rangle \).

Theorem (2)

(i) Let \( W = \{ p + \overline{u} | \overline{u} \in \mathcal{U} \} \) be an affine subspace of \( \mathbb{R}^n \) of dimension \( m \). There exists an \( (n - m) \times n \) matrix \( A \) of rank \( n - m \) s.t.

\[
W = \{ \mathbf{v} \in \mathbb{R}^n | A\mathbf{v} = A\mathbf{p} \}
\]

(ii) Let \( k \leq n \), \( A \) be an \( k \times n \) matrix of rank \( k \) and \( \mathbf{w} \in \mathbb{R}^k \). The set

\[
W = \{ \mathbf{v} \in \mathbb{R}^n | A\mathbf{v} = \mathbf{w} \}
\]

is an affine subspace of \( \mathbb{R}^n \) of dimension \( n - k \).
**Theorem (2)**

(ii) Let \( k \leq n \), \( A \) be an \( k \times n \) matrix of rank \( k \) and \( \vec{w} \in \mathbb{R}^k \). The set

\[
W = \{ \vec{v} \in \mathbb{R}^n | A\vec{v} = \vec{w} \}
\]

is an affine subspace of \( \mathbb{R}^n \) of dimension \( n - k \).

**Proof of (ii):**

The subspace \( U = \text{Null}(A) \) of \( \mathbb{R}^n \) has dimension \( n - k \).

Consider the equation \( A\vec{v} = \vec{w} \): since \( \text{Rank}(A) = k \), it has at least one solution noted \( \vec{p} \);

For any \( \vec{u} \in U \), one has \( A(\vec{p} + \vec{u}) = A\vec{p} = \vec{w} \); the set

\[
W = \{ \vec{v} \in \mathbb{R}^n | A\vec{v} = \vec{w} \}
\]

is an affine subspace of \( \mathbb{R}^n \) of dimension \( n - k \).
Affine subspaces are sols of systems of linear equations

**Theorem (2)**

(i) Let \( W = \{ p + \vec{u} | \vec{u} \in \mathcal{U} \} \) be an affine subspace of \( \mathbb{R}^n \) of dimension \( m \). There exists an \((n - m) \times n\) matrix \( A \) of rank \( n - m \) s.t.

\[
W = \{ v \in \mathbb{R}^n | Av = Ap \}
\]

**Proof of (i):**

\( \mathcal{U} \) is a subspace of \( \mathbb{R}^n \): \( \mathcal{U} \perp \) has dimension \( n - m \), and has a basis \( \langle \vec{v}_1, \ldots, \vec{v}_{n-m} \rangle \).

Let \( A = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_{n-m} \end{bmatrix} \); \( \text{Rank}(A) = n - m \) and \( \text{Null}(A) = \mathcal{U} \). One has:

- Let \( v \in \{ v \in \mathbb{R}^n | Av = Ap \} \): \( Av = Ap \Rightarrow A(v - p) = \vec{0} \Rightarrow v = p + \vec{u} \) where \( \vec{u} \in \mathcal{U} \Rightarrow v \in W 

- Let \( v \in W \): \( \exists \vec{u} \in \mathcal{U} \) s.t. \( v = p + \vec{u} \), \( Av = A(p + \vec{u}) \Rightarrow Av = Ap \)
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Let $W = p + U$ be an affine subspace of $\mathbb{R}^3$. If $\text{Dim}(W) =$

0: $U = \{\vec{0}\}$ (because $\text{Dim}(U) = 0$):

$W = \{p + \vec{0}\}$

$W$ is the point $p$
Affine subspaces of $\mathbb{R}^3$

Let $W = p + U$ be an affine subspace of $\mathbb{R}^3$. If $Dim(W) =$

0: $U = \{\vec{0}\}$ (because $Dim(U) = 0$):
$W = \{p + \vec{0}\}$
$W$ is the point $p$

1: $U = \{a\vec{c}|a \in \mathbb{R}\}$ where $\langle \vec{c} \rangle$ is a basis for $U$ (because $Dim(U) = 1$):
$W = \{p + a\vec{c}|a \in \mathbb{R}\}$
$W$ is the line passing through $p$ and $p + \vec{c}$
Affine subspaces of $\mathbb{R}^3$:

Let $W = p + U$ be an affine subspace of $\mathbb{R}^3$. If $\text{Dim}(W) =$

0: $U = \{\vec{0}\}$ (because $\text{Dim}(U) = 0$):
- $W = \{p + \vec{0}\}$
- $W$ is the point $p$

1: $U = \{a\vec{c}|a \in \mathbb{R}\}$ where $\langle \vec{c}\rangle$ is a basis for $U$ (because $\text{Dim}(U) = 1$):
- $W = \{p + a\vec{c}|a \in \mathbb{R}\}$
- $W$ is the line passing through $p$ and $p + \vec{c}$

2: $U = \{a_1\vec{c}_1 + a_2\vec{c}_2|a_1, a_2 \in \mathbb{R}\}$ where $\langle \vec{c}_1, \vec{c}_2\rangle$ is a basis for $U$ ($\text{Dim}(U) = 2$):
- $W = \{p + a_1\vec{c}_1 + a_2\vec{c}_2|a_1, a_2 \in \mathbb{R}\}$
- $W$ is the plane passing through $u, u + \vec{c}_1, u + \vec{c}_2$
Affine subspaces of $\mathbb{R}^3$

Let $W = p + U$ be an affine subspace of $\mathbb{R}^3$. If $\text{Dim}(W) = \ldots$

0: $U = \{0\}$ (because $\text{Dim}(U) = 0$):
   $W = \{p + 0\}$
   $W$ is the point $p$

1: $U = \{a\bar{c} | a \in \mathbb{R}\}$ where $\langle \bar{c} \rangle$ is a basis for $U$ (because $\text{Dim}(U) = 1$):
   $W = \{p + a\bar{c} | a \in \mathbb{R}\}$
   $W$ is the line passing through $p$ and $p + \bar{c}$

2: $U = \{a_1\bar{c}_1 + a_2\bar{c}_2 | a_1, a_2 \in \mathbb{R}\}$ where $\langle \bar{c}_1, \bar{c}_2 \rangle$ is a basis for $U$ ($\text{Dim}(U) = 2$):
   $W = \{p + a_1\bar{c}_1 + a_2\bar{c}_2 | a_1, a_2 \in \mathbb{R}\}$
   $W$ is the plane passing through $u, u + \bar{c}_1, u + \bar{c}_2$

3: $\text{Dim}(U) = 3$ and $U = \mathbb{R}^3$ and $W = \mathbb{R}^3$
Affine subspaces of $\mathbb{R}^3$:

Let $W = p + U$ be an affine subspace of $\mathbb{R}^3$. If $\text{Dim}(W) =$

0: $U = \{0\}$ (because $\text{Dim}(U) = 0$):

$W = \{p + 0\}$

$W$ is the point $p$ ← parameterized representation ← point representation

1: $U = \{a \vec{c}| a \in \mathbb{R}\}$ where $\langle \vec{c} \rangle$ is a basis for $U$ (because $\text{Dim}(U) = 1$):

$W = \{p + a \vec{c}| a \in \mathbb{R}\}$

$W$ is the line passing through $p$ and $p + \vec{c}$ ← parameterized representation ← point representation

2: $U = \{a_1 \vec{c}_1 + a_2 \vec{c}_2| a_1, a_2 \in \mathbb{R}\}$ where $\langle \vec{c}_1, \vec{c}_2 \rangle$ is a basis for $U$ ($\text{Dim}(U) = 2$):

$W = \{p + a_1 \vec{c}_1 + a_2 \vec{c}_2| a_1, a_2 \in \mathbb{R}\}$

$W$ is the plane passing through $u, u + \vec{c}_1, u + \vec{c}_2$ ← parameterized representation ← point representation

3: $\text{Dim}(U) = 3$ and $U = \mathbb{R}^3$ and $W = \mathbb{R}^3$
Affine subspaces of $\mathbb{R}^3$

Let $W = p + U$ be an affine subspace of $\mathbb{R}^3$. If $\text{Dim}(W) =$

0: $U = \{0\}$ (because $\text{Dim}(U) = 0$):

$W = \{p + \vec{0}\}$

$W$ is the point $p$

$W = \{v \in \mathbb{R}^3 | Av = Ap\}$ where $A$ is $3 \times 3$ and $\text{Rank}(A) = 3$ ($\leftarrow$ implicit rep.)

1: $U = \{a\vec{c}|a \in \mathbb{R}\}$ where $\langle \vec{c} \rangle$ is a basis for $U$ (because $\text{Dim}(U) = 1$):

$W = \{p + a\vec{c}|a \in \mathbb{R}\}$

$W$ is the line passing through $p$ and $p + \vec{c}$

$W = \{v \in \mathbb{R}^3 | Av = Ap\}$ where $A$ is $2 \times 3$ and $\text{Rank}(A) = 2$ ($\leftarrow$ implicit rep.)

2: $U = \{a_1\vec{c}_1 + a_2\vec{c}_2|a_1, a_2 \in \mathbb{R}\}$ where $\langle \vec{c}_1, \vec{c}_2 \rangle$ is a basis for $U$ ($\text{Dim}(U) = 2$):

$W = \{p + a_1\vec{c}_1 + a_2\vec{c}_2|a_1, a_2 \in \mathbb{R}\}$

$W$ is the plane passing through $u, u + \vec{c}_1, u + \vec{c}_2$

$W = \{v \in \mathbb{R}^3 | Av = Ap\}$ where $A$ is $1 \times 3$ and $\text{Rank}(A) = 1$ ($\leftarrow$ implicit rep.)

Theorem 2 $\Rightarrow \exists$ an $(3 - \text{Dim}(W)) \times 3$ matrix $A$ of rank $3 - \text{Dim}(W)$ s.t.

$\vec{v} \in W \iff A\vec{v} = A\vec{p}$
Lines of $\mathbb{R}^3$

Let $L = p + U$ be a line of $\mathbb{R}^3$ (i.e. $\text{Dim}(U) = 1$).

Point representation: $L$ is the line passing through $a, b$, where $a \neq b$. 

Lines of $\mathbb{R}^3$

Let $L = \mathbf{p} + \mathcal{U}$ be a line of $\mathbb{R}^3$ (i.e. $\text{Dim}(\mathcal{U}) = 1$).

**Point representation:** $L$ is the line passing through $\mathbf{a}, \mathbf{b}$, where $\mathbf{a} \neq \mathbf{b}$.

**Parameterized representation:** $L = \{ \mathbf{a} + t(\mathbf{b} - \mathbf{a}) | t \in \mathbb{R} \}$

The subspace $\mathcal{U}$ associated to $L$ has basis $\langle \mathbf{b} - \mathbf{a} \rangle$. 

Lines of $\mathbb{R}^3$

Let $L = p + U$ be a line of $\mathbb{R}^3$ (i.e. $\text{Dim}(U) = 1$).

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**Parameterized representation:** $L = \{a + t(b - a) | t \in \mathbb{R}\}$

The subspace $U$ associated to $L$ has basis $\langle b - a \rangle$.

**Implicit representation:** according to the proof of (i) of thm. 2:

let $\langle \vec{v}_1, \vec{v}_2 \rangle$ be the basis for an orthogonal complement of $U$,

and $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$ be a $2 \times 3$ matrix with $\text{Rank}(A) = 2$.

Then

$$L = \{v \in \mathbb{R}^3 | Av = Aa\}$$

is an implicit representation for $L$. 

---

Rémi Imbach

March 12, 2019
Lines of $\mathbb{R}^3$

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Then

$$L = \{v \in \mathbb{R}^3 | Av = Aa\}$$

is an implicit representation for $L$.

How to find $\langle \vec{v}_1, \vec{v}_2 \rangle$? call $\text{basisForOrthoComp}(\langle b - a \rangle, S)$

(where $S$ is the standard basis for $\mathbb{R}^3$).
Planes of $\mathbb{R}^3$

Let $P = p + \mathcal{U}$ be a plane of $\mathbb{R}^3$ \textit{(i.e. Dim}(P) = 2). 

\textbf{Point representation}: $P$ is the plane passing through $d, e, f$, where $d, e, f$ are pairwise disjoint and not on the same line.
Planes of $\mathbb{R}^3$

Let $P = p + \mathcal{U}$ be a plane of $\mathbb{R}^3$ (i.e. $\text{Dim}(P) = 2$).

**Point representation:** $P$ is the plane passing through $d, e, f$, where $d, e, f$ are pairwise disjoint and not on the same line.

**Parameterized representation:** $P = \{d + t_1(e - d) + t_2(f - d) | t_1, t_2 \in \mathbb{R}\}$

The subspace $\mathcal{U}$ associated to $P$ has basis $\langle e - d, f - d \rangle$. 
Planes of $\mathbb{R}^3$

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The subspace $\mathcal{U}$ associated to $P$ has basis $\langle e - d, f - d \rangle$.

**Implicit representation:** according to the proof of (i) of thm. 2:
let $\langle \vec{v} \rangle$ be the basis for an orthogonal complement of $\mathcal{U}$, and $A = [\vec{v}]$ be a $1 \times 3$ matrix with $\text{Rank}(A) = 1$.
Then

$$P = \{v \in \mathbb{R}^3 | Av = Ad\}$$

is an implicit representation for $P$. 
Planes of $\mathbb{R}^3$

Let $P = p + \mathcal{U}$ be a plane of $\mathbb{R}^3$ (i.e. $\text{Dim}(P) = 2$).

Point representation: $P$ is the plane passing through $d, e, f$, where $d, e, f$ are pairwise disjoint and not on the same line.

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and $A = [\vec{v}]$ be a $1 \times 3$ matrix with $\text{Rank}(A) = 1$.
Then
$$P = \{v \in \mathbb{R}^3 | Av = Ad\}$$
is an implicit representation for $P$.

How to find $\langle \vec{v} \rangle$? call $\text{basisForOrthoComp}(\langle e - d, f - d \rangle, S)$ or
The cross product

Definition: Let \( \vec{u} = \langle u[1], u[2], u[3] \rangle \) and \( \vec{v} = \langle v[1], v[2], v[3] \rangle \) be two vectors of \( \mathbb{R}^3 \). The cross product of \( \vec{u} \) and \( \vec{v} \), noted \( \vec{u} \times \vec{v} \), is defined as the vector of \( \mathbb{R}^3 \):

\[
\begin{bmatrix}
\end{bmatrix}
\]

Proposition: If \( \vec{u} \) and \( \vec{v} \), are non-zero and not colinear,

(i) \( \vec{u} \times \vec{v} \neq \vec{0} \)

(ii) \((\vec{u} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{v}) \cdot \vec{v} = \vec{0} \)

(iii) \( \vec{u} \times \vec{v} \) is a basis for the orthogonal complement of \( \text{Span}(\langle \vec{u}, \vec{v} \rangle) \)
Intersection of a line and a plane

Let \( a, b \) be two points s.t. \( a \neq b \) and

\[
L = \{ a + t(b - a) | t \in \mathbb{R} \}
\]

Let \( d, e, f \) be three points pairwise disjoint and not in the same line and

\[
P = \{ d + t_1(e - d) + t_2(f - d) | t_1, t_2 \in \mathbb{R} \}
\]

When do \( L \) and \( P \) have:

- no intersection?
- a line as intersection?
- a unique intersection point?
Intersection of a line and a plane

Let \( a, b \) be two points s.t. \( a \neq b \) and

\[
L = \{a + t(b - a) | t \in \mathbb{R}\}
\]

Let \( d, e, f \) be three points pairwise disjoint and not in the same line and

\[
P = \{d + t_1(e - d) + t_2(f - d) | t_1, t_2 \in \mathbb{R}\}
\]

When do \( L \) and \( P \) have:

- no intersection? \( \text{When } (b - a) \in \text{Span}(\langle e - d, f - d \rangle) \) and \( a \notin P \)
- a line as intersection? \( \text{When } (b - a) \in \text{Span}(\langle e - d, f - d \rangle) \) and \( a \in P \)
- a unique intersection point? \( \text{When } (b - a) \notin \text{Span}(\langle e - d, f - d \rangle) \)
Intersection of a line and a plane

Let \( a, b \) be two points s.t. \( a \neq b \) and

\[
L = \{a + t(b - a) | t \in \mathbb{R}\} = \{v \in \mathbb{R}^3 | Av = Aa\}
\]

Let \( d, e, f \) be three points pairwise disjoint and not in the same line and

\[
P = \{d + t_1(e - d) + t_2(f - d) | t_1, t_2 \in \mathbb{R}\} = \{v \in \mathbb{R}^3 | Bv = Bd\}
\]

When do \( L \) and \( P \) have:

- no intersection? When \( (b - a) \in \text{Span}(\langle e - d, f - d \rangle) \) and \( a \notin P \)
- a line as intersection? When \( (b - a) \in \text{Span}(\langle e - d, f - d \rangle) \) and \( a \in P \)
- a unique intersection point? When \( (b - a) \notin \text{Span}(\langle e - d, f - d \rangle) \)

In the latter case, the intersection is the unique solution of

\[
\begin{bmatrix}
A[1,:]
A[2,:]
B[1,:]
\end{bmatrix}
\begin{bmatrix}
v
\end{bmatrix}
= \begin{bmatrix}
Aa[1]
Aa[2]
Bd[1]
\end{bmatrix}
\]
Intersection of a line and a plane

Procedure: IntersectLinePlane( (a, b), (d, e, f) )

input: (a, b) two points s.t. a ≠ b,
       (d, e, f) three points pairwise disjoint and not in the same line
       s.t. (b − a) /∈ Span(⟨e − d, f − d⟩)
output: the intersection of the line passing through a, b
        with the plane passing through d, e, f

01. ⟨v_1, v_2⟩ ← basisForOrthoComp(⟨b − a⟩, S)
02. M ← \[
\begin{bmatrix}
  v_1 \\
v_2
\end{bmatrix},
\tilde{y} ← Ma
\]
03. ⟨v_3⟩ ← (e − d) × (f − d)
04. M ← \[
\begin{bmatrix}
v_3
\end{bmatrix},
\tilde{y}' ← Md
\]
05. M ← \[
\begin{bmatrix}
  v_1 \\
v_2 \\
v_3
\end{bmatrix},
\tilde{y} ← \[
\begin{bmatrix}
  \tilde{y}[1] \\
  \tilde{y}[2] \\
  \tilde{y}'[1]
\end{bmatrix}
\]
06. return the unique solution of M\vec{x} = \tilde{y}
Course 6 - Affine geometry

I The ray-tracing algorithm
II Additional material in linear algebra
III Affine subspaces of $\mathbb{R}^n$
IV Systems of coordinates
   1 Systems of coordinates
   2 Change of systems
V Intersect ray patch procedure
**Systems of coordinates**

Let $W = p + U$ be an affine subspace of $\mathbb{R}^n$.

**Definition:** [System of coordinates of an affine subspace]

A *system of coordinates* for $W$ is a couple $C = (c, C_U)$ where $c$ is a point of $W$, and $C_U$ is a basis for $U$.

If $C_U$ is orthogonal, $C = (c, C_U)$ is said *orthogonal*.
If $C_U$ is orthonormal, $C = (c, C_U)$ is said *orthonormal*. 
Systems of coordinates

Let $W = p + U$ be an affine subspace of $\mathbb{R}^n$.

Definition: [System of coordinates of an affine subspace]
A system of coordinates for $W$ is a couple $C = (c, C_U)$ where $c$ is a point of $W$, and $C_U$ is a basis for $U$.

If $C_U$ is orthogonal, $C = (c, C_U)$ is said orthogonal.
If $C_U$ is orthonormal, $C = (c, C_U)$ is said orthonormal.

Proposition:
Let $C = (c, C_U)$ be a coordinate system for $W$, where $C_U = \langle \vec{c}_1, \ldots, \vec{c}_m \rangle$.
For any $v \in W$ there exists a unique tuple $\langle v_1, \ldots, v_m \rangle$ of real numbers such that

$$v = c + v_1\vec{c}_1 + v_2\vec{c}_2 + \ldots + v_m\vec{c}_m$$

Definition: [Coordinates of a point]
Let $C = (c, C_U)$ be a coordinate system for $W$, and $v \in W$. The unique tuple of the above proposition is noted $\text{Coords}(v, C)$ and called coordinates of $v$ in $C$. 
Examples

- \( S = (0, \langle \vec{e}^1, \ldots, \vec{e}^n \rangle) \) is the \textit{standard} system of coordinates for \( \mathbb{R}^n \), it is orthonormal
Examples

- \( S = (0, \langle \vec{e}^1, \ldots, \vec{e}^n \rangle) \) is the *standard* system of coordinates for \( \mathbb{R}^n \), it is orthonormal
- \( C = (c, \langle \vec{x}_c, \vec{y}_c, \vec{z}_c \rangle) \) is an orthonormal coordinate system for \( \mathbb{R}^3 \)
Examples

- $S = (0, \langle \vec{e}^1, \ldots, \vec{e}^n \rangle)$ is the **standard** system of coordinates for $\mathbb{R}^n$, it is orthonormal

- $C = (c, \langle \vec{x}_c, \vec{y}_c, \vec{z}_c \rangle)$ is an orthonormal coordinate system for $\mathbb{R}^3$

  $\text{Coords}(\text{pix}(1, 1), C) = \left\langle -\frac{w}{2} + \frac{w}{2r_w}, -\frac{h}{2} + \frac{f}{2r_h}, f \right\rangle$
Examples

- $S = (0, \langle \vec{e}^1, \ldots, \vec{e}^n \rangle)$ is the *standard* system of coordinates for $\mathbb{R}^n$, it is orthonormal

- $C = (c, \langle \vec{x}_c, \vec{y}_c, \vec{z}_c \rangle)$ is an orthonormal coordinate system for $\mathbb{R}^3$

  \[
  \text{Coords}(\text{pix}(1,1), C) = \left\langle -\frac{w}{2} + \frac{w}{2rw}, -\frac{h}{2} + \frac{f}{2rh}, f \right\rangle
  \]

  \[
  \text{Coords}(\text{pix}(i,j), C) = \left\langle -\frac{w}{2} + \frac{(i-1)w}{rw} + \frac{w}{2rw}, -\frac{h}{2} + \frac{(j-1)h}{rh} + \frac{h}{2rh}, f \right\rangle
  \]
Examples

- \( S = (0, \langle \vec{e}^1, \ldots, \vec{e}^n \rangle) \) is the standard system of coordinates for \( \mathbb{R}^n \), it is orthonormal

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  \]
  
  \[
  \text{Coords}(\text{pix}(i, j), C) = \langle -\frac{w}{2} + \frac{(i-1)w}{r_w} + \frac{w}{2r_w}, -\frac{h}{2} + \frac{(j-1)h}{r_h} + \frac{h}{2r_h}, f \rangle
  \]

- \( F = (p_1, \langle (p_2 - p_1), (p_3 - p_1) \rangle) \) is a coordinate system for the plane passing through \( p_1, p_2, p_3 \)
Examples

- $S = (0, \langle \vec{e}^1, \ldots, \vec{e}^n \rangle)$ is the standard system of coordinates for $\mathbb{R}^n$, it is orthonormal.

- $C = (c, \langle \vec{x}_c, \vec{y}_c, \vec{z}_c \rangle)$ is an orthonormal coordinate system for $\mathbb{R}^3$
  
  \[
  \text{Coords}(\text{pix}(1, 1), C) = \langle -\frac{w}{2} + \frac{w}{2r_w}, -\frac{h}{2} + \frac{f}{2r_h}, f \rangle
  \]
  
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  \text{Coords}(\text{pix}(i, j), C) = \langle -\frac{w}{2} + \frac{(i-1)w}{r_w} + \frac{w}{2r_w}, -\frac{h}{2} + \frac{(j-1)h}{r_h} + \frac{h}{2r_h}, f \rangle
  \]

- $F = (p_1, \langle (p_2 - p_1), (p_3 - p_1) \rangle)$ is a coordinate system for the plane passing through $p_1, p_2, p_3$
  
  \[
  \text{Coords}(p_2, F) = \langle 1, 0 \rangle
  \]
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The problem

Let $W = \mathbf{p} + \mathcal{U}$ be an affine subspace of $\mathbb{R}^n$, and $C = (\mathbf{c}, C_\mathcal{U})$, be a coordinate system for $W$ with $C_\mathcal{U} = \langle \mathbf{c}_1, \ldots, \mathbf{c}_m \rangle$. Let $\mathbf{a}$ be a point of $W$.

Recall: $S = (\mathbf{0}, S_n)$ where $S_n = \langle \mathbf{e}^1, \ldots, \mathbf{e}^n \rangle$ is the standard coordinate system of $\mathbb{R}^n$.

How to obtain $\text{Coords}(\mathbf{a}, S)$ from $\text{Coords}(\mathbf{a}, C)$?
The problem

Let $W = \mathbf{p} + \mathcal{U}$ be an affine subspace of $\mathbb{R}^n$, and $\mathcal{C} = (\mathbf{c}, C_U)$, be a coordinate system for $W$ with $C_U = \langle \vec{c}_1, \ldots, \vec{c}_m \rangle$. Let $\mathbf{a}$ be a point of $W$.

Recall: $S = (\mathbf{0}, S_n)$ where $S_n = \langle \vec{e}^1, \ldots, \vec{e}^n \rangle$ is the standard coordinate system of $\mathbb{R}^n$.

How to obtain $\text{Coords}(\mathbf{a}, S)$ from $\text{Coords}(\mathbf{a}, \mathcal{C})$?

Let $\text{Coords}(\mathbf{a}, \mathcal{C}) = \langle a_1, \ldots, a_m \rangle$; then $\mathbf{a} = \mathbf{c} + a_1 \vec{c}_1 + \ldots + a_m \vec{c}_m$

Since $\vec{c}_1, \ldots, \vec{c}_m$ are given in the standard basis, $\mathbf{c} + a_1 \vec{c}_1 + \ldots + a_m \vec{c}_m$ is the tuple of coordinates of $\mathbf{a}$ in $S$. 
Application: get the coordinates of the \((i,j)\)-th pixel

Global variables: \( C = (c, (\vec{x}_c, \vec{y}_c, \vec{z}_c)), f, w, h, r_w, r_h, \) the scene

Procedure: \( \text{CoordsOfPixInStandard}(i,j) \)

**input:** \( i,j \) the pair of indexes of a pixel  
**output:** \( \text{Coords}(\text{pix}(i,j), S) \)

01. return \( c + \left(-\frac{w}{2} + \frac{(i-1)w}{r_w} + \frac{w}{2r_w}\right)\vec{x}_c + \left(-\frac{h}{2} + \frac{(j-1)h}{r_h} + \frac{h}{2r_h}\right)\vec{y}_c + f\vec{z}_c \)
The problem

Let $W = p + U$ be an affine subspace of $\mathbb{R}^n$, and $C = (c, C_U)$, be a coordinate system for $W$ with $C_U = \langle \vec{c}_1, \ldots, \vec{c}_m \rangle$. Let $a$ be a point of $W$.

Recall: $S = (0, S_n)$ where $S_n = \langle \vec{e}_1, \ldots, \vec{e}_n \rangle$ is the standard coordinate system of $\mathbb{R}^n$.

How to obtain $\text{Coords}(a, S)$ from $\text{Coords}(a, C)$?
Let $\text{Coords}(a, C) = \langle a_1, \ldots, a_m \rangle$; then $a = c + a_1 \vec{c}_1 + \ldots + a_m \vec{c}_m$
Since $\vec{c}_1, \ldots, \vec{c}_m$ are given in the standard basis, $c + a_1 \vec{c}_1 + \ldots + a_m \vec{c}_m$ is the tuple of coordinates of $a$ in $S$.

How to obtain $\text{Coords}(a, C)$ from $\text{Coords}(a, S)$?
From $\text{Coords}(a, S)$ to $\text{Coords}(a, C)$

$C = (c, C_U)$ with $C_U = \langle \vec{c}_1, \ldots, \vec{c}_m \rangle$.

$a, c \in U$ are given in the standard system of coordinates:

$a = \text{Coords}(a, S)$ and $c = \text{Coords}(c, S)$.

Suppose $\langle a_1, \ldots, a_m \rangle = \text{Coords}(a, C)$

Then

$$a = c + a_1 \vec{c}_1 + \ldots + a_m \vec{c}_m$$

and

$$a - c = a_1 \vec{c}_1 + \ldots + a_m \vec{c}_m$$

i.e. $\langle a_1, \ldots, a_m \rangle = \text{Coords}(a - c, C_U)$

Let $C$ be the $n \times m$ matrix $[\vec{c}_1, \ldots, \vec{c}_m]$, and consider the system:

$$C \vec{x} = (a - c)$$

Since $\text{Rank}(C) = m$, $\text{Dim(Null}(C)) = 0$ (system of category III) and $(a - c) \in \text{Im}(C)$,

$$C \vec{x} = (a - c)$$

has as unique solution $\text{Coords}(a, C)$.
Application: does a point lie in a triangle

Let \( \mathbf{d}, \mathbf{e}, \mathbf{f} \) be three points in \( \mathbb{R}^3 \) not in the same line, and \( P \) be the plane passing through \( \mathbf{d}, \mathbf{e}, \mathbf{f} \):

\[
P = \{ \mathbf{d} + t_1(\mathbf{e} - \mathbf{d}) + t_2(\mathbf{f} - \mathbf{d}) | t_1, t_2 \in \mathbb{R} \}
\]

\( C = (\mathbf{d}, \langle \mathbf{e} - \mathbf{d}, \mathbf{f} - \mathbf{d} \rangle) \) is a coordinate system for \( P \).

Let \( \mathbf{p} \) be a point of \( P \).

**Question 1:** Find the coordinates of \( \mathbf{p} \) in \( C \):

**Question 2:** Let \( \text{Coords}(\mathbf{p}, C) = \langle a_1, a_2 \rangle \).

Give a condition on \( a_1, a_2 \) for \( \mathbf{p} \) to lie inside the triangle of vertices \( \mathbf{d}, \mathbf{e}, \mathbf{f} \).
Application: does a point lie in a triangle

Let \(d, e, f\) be three points in \(\mathbb{R}^3\) not in the same line, and \(P\) be the plane passing through \(d, e, f\):

\[
P = \{d + t_1(e - d) + t_2(f - d) | t_1, t_2 \in \mathbb{R}\}
\]

\(C = (d, \langle e - d, f - d \rangle)\) is a coordinate system for \(P\).

Let \(p\) be a point of \(P\).

**Question 1:** Find the coordinates of \(p\) in \(C\):

\(\text{Coords}(p, C)\) is the unique solution of the system

\[
[e - d \quad f - d] \bar{x} = (p - d)
\]

**Question 2:** Let \(\text{Coords}(p, C) = \langle a_1, a_2 \rangle\).

Give a condition on \(a_1, a_2\) for \(p\) to lie inside the triangle of vertices \(d, e, f\).

\(p\) lies in the triangle if and only if \(a_1 \geq 0, a_2 \geq 0, \text{ and } |a_1 + a_2| \leq 1.\)
Application: does a point lie on an half-line

Let \(a, b\) be two distinct points of \(\mathbb{R}^3\) and \(L\) be the line passing through \(a, b\):

\[
L = \{a + t(b - a) | t \in \mathbb{R}\}
\]

\(L\) is an affine subspace of \(\mathbb{R}^3\)

\(C = (a, \langle b - a \rangle)\) is a coordinate system for \(L\).

Let \(p\) be a point of \(L\).

**Question 1:** Find the coordinates of \(p\) in \(C\):

**Question 2:** Let \(\text{Coords}(p, C) = \langle a \rangle\).

Give a condition on \(a\) for \(p\) to lie in the half-line \([a, b)\).
Procedures

Procedure: IsPointInTriangle(p, d, e, f)

input: (d, e, f) three points not in the same line
(p) a point in the plane passing through d, e, f
output: true if p lies in the triangle of vertices d, e, f, false otherwise.

01. \bar{v} \leftarrow \text{the unique solution of } \begin{bmatrix} e - d & f - d \end{bmatrix} \bar{x} = (p - d)
02. if \bar{v}[1] \geq 0 \text{ and } \bar{v}[2] \geq 0 \text{ and } |\bar{v}[1] + \bar{v}[2]| \leq 1 \text{ then }
03. \text{return true }
04. endif
05. \text{return false }

Procedure: IsPointInHalfLine(p, a, b)

input: (a, b) two distinct points
(p) a point in the line passing through a, b
output: true if p lies in the half-line a, b, false otherwise.

01. ...
Course 6 - Affine geometry

I The ray-tracing algorithm
II Additional material in linear algebra
III Affine subspaces of $\mathbb{R}^n$
IV Systems of coordinates
V Intersect ray patch procedure
Intersect ray patch procedure

Global variables: $C = (c, (\tilde{x}_c, \tilde{y}_c, \tilde{z}_c))$, $f$, $w$, $h$, $r_w$, $r_h$, the scene

Procedure: IntersectRayPatch($(i,j)$, $k$)

input: $(i, j)$ the index of a pixel
         $k$ the index of a patch
output: if the ray passing through the $(i, j)$-th pixel intersects the $k$-th patch,
        returns the distance between $c$ and that intersection
        otherwise $+\infty$

01. $b \leftarrow \text{CoordsOfPixInStandard}(i,j)$
02. $d, e, f \leftarrow \text{vertices of the } k\text{-th patch}$
03. $p \leftarrow \text{IntersectLinePlane}( (c, b), (d, e, f) )$
04. if IsPointInTriangle($p, d, e, f$) and IsPointInHalfLine($p, c, b$) then
05. return $\|p - c\|_2$
06. endif
07. return $+\infty$