Mathematical Techniques for Computer Science Applications

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webpage:
https://cs.nyu.edu/courses/spring19/CSCI-GA.1180-001/
Course 3 - Matrices and linear maps

I Definitions and examples
   1 First examples
   2 More definitions

II Operations and properties

III Linear maps

IV Matrices in Matlab
Definition

A \( m \times n \) matrix is a two dimensional array of real numbers, with \( m \) rows and \( n \) columns.

\[
A = \begin{bmatrix}
1 & 2.1 & 3.1 \\
2.1 & 1.2 & 5
\end{bmatrix}
\] is a \( 2 \times 3 \) matrix,

\[
B = \begin{bmatrix}
1.2 & 4.1 \\
2.5 & 5
\end{bmatrix}
\] is a \( 2 \times 2 \) matrix.

A matrix is \emph{square} when \( m = n \) (nb of rows=nb of columns).

If \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), \( A[i,j] \) will denote the elements of \( A \) at row \( i \) and column \( j \). (It is more often noted \( A_{i,j} \) or \( A_{ij} \)).

Here \( A[2,1] = 2.1 \).
First examples

- Black & White image of \( n \times m \) pixels: \( m \times n \) matrix with 0 and 1
First examples

• Black & White image of \( n \times m \) pixels: \( m \times n \) matrix with 0 and 1

• Database of the online store with \( n \) products and \( m \) customers: \( m \times n \) matrix with integers

**On-line store:** 5 products \( p_1, p_2, \ldots, p_5 \),
5 customers \( c_1, c_2, \ldots, c_5 \).

**Database:** 2-dimensional array

<table>
<thead>
<tr>
<th></th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( p_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>4</td>
<td>8</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Value at line \( i \) and column \( j \) is \( k \): \( c_i \) bought \( k \) items of product \( p_j \).
First examples

- Black & White image of $n \times m$ pixels: $m \times n$ matrix with 0 and 1
- Database of the online store with $n$ products and $m$ customers: $m \times n$ matrix with integers
  
  above, “matrix” is just another word for “2D table”
First examples

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• Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$: uniquely characterized by an $m \times n$ matrix

Let $\vec{u} \in \mathbb{R}^3$, and $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ defined as:

$$f_1(\vec{u}) = 3\vec{u}[1] - 5\vec{u}[2] + 2\vec{u}[3]$$
$$f_2(\vec{u}) = 2\vec{u}[1] + 4\vec{u}[2] - 3\vec{u}[3]$$
First examples

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Let $\vec{u} \in \mathbb{R}^3$, and $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ defined as:

\[
\begin{align*}
  f_1(\vec{u}) &= 3\vec{u}[1] - 5\vec{u}[2] + 2\vec{u}[3] & \leftarrow \text{linear, characterized by } \vec{v}_1 = \langle 3, -5, 2 \rangle \\
  f_2(\vec{u}) &= 2\vec{u}[1] + 4\vec{u}[2] - 3\vec{u}[3]
\end{align*}
\]
First examples

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  f_2(\vec{u}) &= 2\vec{u}[1] + 4\vec{u}[2] - 3\vec{u}[3] \quad \leftarrow \text{linear, characterized by } \vec{v}_2 = \langle 2, 4, -3 \rangle
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\end{align*}
\]

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined as: $F(\vec{u}) = \langle f_1(\vec{u}), f_2(\vec{u}) \rangle$
First examples

• Black & White image of \( n \times m \) pixels: \( m \times n \) matrix with 0 and 1

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• Linear maps from \( \mathbb{R}^n \) to \( \mathbb{R}^m \): uniquely characterized by an \( m \times n \) matrix

Let \( \vec{u} \in \mathbb{R}^3 \), and \( f_1, f_2 : \mathbb{R}^3 \to \mathbb{R} \) defined as:

\[
\begin{align*}
    f_1(\vec{u}) &= 3\vec{u}[1] - 5\vec{u}[2] + 2\vec{u}[3] & \leftarrow \text{linear, characterized by } \vec{v}_1 = \langle 3, -5, 2 \rangle \\
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\end{align*}
\]

Let \( F : \mathbb{R}^3 \to \mathbb{R}^2 \) defined as: \( F(\vec{u}) = \langle f_1(\vec{u}), f_2(\vec{u}) \rangle = \begin{bmatrix} f_1(\vec{u}) \\ f_2(\vec{u}) \end{bmatrix} \)
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- Black & White image of $n \times m$ pixels: $m \times n$ matrix with 0 and 1
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$F$ is characterized by $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 2 \\ 2 & 4 & -3 \end{bmatrix}$
First examples

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• Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$: uniquely characterized by an $m \times n$ matrix
  • some geometric transformations (computer graphics)
  • linear operations in signal processing
  • Markov chains (game theory, AI, . . . )
  • . . .
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  - some geometric transformations (computer graphics)
  - linear operations in signal processing
  - Markov chains (game theory, AI, ...)
  - ...
- adjacency matrix of a graph
Example: adjacency matrix of a graph

Graph $G$: couple $(V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $E$ is a subset of $V \times V$. Example:

- vertices: $V = \{1, 2, \ldots, 6\}$
- edges: $E = \{(1, 2), (2, 3), \ldots, (6, 1)\}$
Example: adjacency matrix of a graph

Graph $G$: couple $(V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $E$ is a subset of $V \times V$.

Example:

- vertices: $V = \{1, 2, \ldots, 6\}$
- edges: $E = \{(1, 2), (2, 3), \ldots, (6, 1)\}$

Adjacency matrix $G$ of $G$: $n \times n$ matrix defined as:

- $G[i, j] = 1$ if there is an edge from $v_i$ to $v_j$ in $G$,
- $G[i, j] = 0$ otherwise.

$$
G = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 1 & 1 & 1 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 1 & 0 \\
5 & 1 & 0 & 0 & 1 & 0 \\
6 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$
Course 3 - Matrices and linear maps

I Definitions and examples
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Peculiar matrices

$0_{m \times n}$ or 0 is the matrix which all elements are 0.

$0_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_m$ or $I$, called the identity matrix is the square matrix of size $m \times m$ which all elements are 0 but the elements of the diagonal that are 1.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A matrix $D$ is diagonal if it is square and all its elements are 0 but the elements of the diagonal. (Formally: $D$ of size $m \times m$ is diagonal iff $i \neq j \Rightarrow D[i, j] = 0$)

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$
Rows, columns and vectors

We will call

- \( n \)-dimensional \textit{row vector} a \( 1 \times n \) matrix, and
- \( m \)-dimensional \textit{column vector} a \( m \times 1 \) matrix.

\[ A_1 = \begin{bmatrix} 1 & 2.1 & 3.1 \end{bmatrix} \] is a 3-dimensional row vector

\[ A'_2 = \begin{bmatrix} 2.1 \\ 1.2 \end{bmatrix} \] is a 2-dimensional column vector

An \( m \times n \) matrix can be seen as

- \( m \ n \)-dimensional row vectors, (note \( A[i,:] \) the \( i \)-th row vector of \( A \))
- \( n \ m \)-dimensional column vectors, (note \( A[:,j] \) the \( j \)-th column vector of \( A \)).
Example

\[ A = \begin{bmatrix} 1 & 2.1 & 3.1 \\ 2.1 & 1.2 & 5 \end{bmatrix} \] is a \(2 \times 3\) matrix,

\[ = \begin{bmatrix} A[1, :] \\ A[2, :] \end{bmatrix} \text{ where } A[1, :] = \begin{bmatrix} 1 & 2.1 & 3.1 \end{bmatrix} \text{ and } A[2, :] = \begin{bmatrix} 2.1 & 1.2 & 5 \end{bmatrix} \]

\[ = \begin{bmatrix} A[:, 1] & A[:, 2] & A[:, 3] \end{bmatrix} \text{ where } A[:, 1] = \begin{bmatrix} 1 \\ 2.1 \end{bmatrix}, \ A[:, 2] = \begin{bmatrix} 2.1 \\ 1.2 \end{bmatrix} \text{ and } A[:, 3] = \begin{bmatrix} 3.1 \\ 5 \end{bmatrix} \]
Course 3 - Matrices and linear maps

I  Definitions and examples
II  Operations and properties
   1  Transpose
   2  Scalar multiplication
   3  Sum, difference
   4  Matrix times a vector
   5  Matrix times a matrix
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IV Matrices in Matlab
Transpose

Let $A$ be an $m \times n$ matrix.
The **transpose** of $A$, noted $A^t$, is the $n \times m$ matrix whose rows are the cols of $A$:

$$\forall 1 \leq i \leq n, \forall 1 \leq j \leq m, A^t[i, j] = A[j, i]$$

$$A = \begin{bmatrix} 1 & 2.1 & 3.1 \\ 2.1 & 1.2 & 5 \end{bmatrix}, \ A^t = \begin{bmatrix} 1 & 2.1 \\ 2.1 & 1.2 \\ 3.1 & 5 \end{bmatrix}$$
Transpose

Let $A$ be an $m \times n$ matrix. The *transpose* of $A$, noted $A^t$, is the $n \times m$ matrix whose rows are the cols of $A$:

$$ \forall 1 \leq i \leq n, \forall 1 \leq j \leq m, A^t[i, j] = A[j, i] $$

$$ A = \begin{bmatrix} 1 & 2.1 & 3.1 \\ 2.1 & 1.2 & 5 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 2.1 \\ 2.1 & 1.2 \\ 3.1 & 5 \end{bmatrix} $$

In particular:

- the transpose of a $n$-dim. row vector is a $n$-dim. column vector
- the transpose of a $m$-dim. column vector is a $m$-dim. row vector
- if $D$ is a diagonal matrix, $D^t = D.$

$$ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} $$
Properties

Let $A, B, C$ be $m \times n$ matrices, and $r, s \in \mathbb{R}^n$.

(P1) **Self inverse of transposition**: $(A^t)^t = A$
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Scalar multiplication

Let $A$ be an $m \times n$ matrix and $r \in \mathbb{R}$.

The scalar multiplication of $A$ by $r$, noted $rA$ or $rA$ or $Ar$ is the $m \times n$ matrix obtained by multiplying each element of $A$ by $r$:

$$\forall 1 \leq i \leq m, \forall 1 \leq j \leq n, (rA)[i,j] = r(A[i,j])$$

$$A = \begin{bmatrix} 1 & 2.1 & 3.1 \\ 2.1 & 1.2 & 5 \end{bmatrix}, \quad r = 2$$

$$= \begin{bmatrix} 2 \times 1 & 2 \times 2.1 & 2 \times 3.1 \\ 2 \times 2.1 & 2 \times 1.2 & 2 \times 5 \end{bmatrix}$$
Properties

Let $A, B, C$ be $m \times n$ matrices, and $r, s \in \mathbb{R}^n$.

(P1) Self inverse of transposition: $(A^t)^t = A$

(P2) Commutativity of $\cdot$: $rA = Ar$

(P3) Associativity of $\cdot$: $r(sA) = (rs)A$

(P4) $(rA)^t = r(A^t)$

(P5) Commutativity of $+$: $A + B = B + A$

(P6) Associativity of $+$: $(A + B) + C = A + (B + C)$

(P7) Distributivity of $\cdot$ over $+$: $r(A + B) = rA + rB$

(P8) Distributivity of $+$ over $\cdot$: $(r + s)A = rA + sA$

(P9) Distributivity of transposition over $+$: $(A + B)^t = A^t + B^t$

(P10) 0 is the identity for $+$: $A + 0 = 0 + A = A$
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Addition

Let $A, B$ be $m \times n$ matrices.

The *sum* of $A$ and $B$, noted $A + B$, is the $m \times n$ matrix obtained by adding $A$ and $B$ component-wise:

\[
\forall 1 \leq i \leq m, \forall 1 \leq j \leq n, (A + B)[i, j] = A[i, j] + B[i, j]
\]

The addition is defined only if $A$ and $B$ have the same dimensions!

- if $A = \begin{bmatrix} 1 & 2.1 & 3.1 \\ 2.1 & 1.2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1.2 & 4.1 \\ 2.5 & 5 \end{bmatrix}$, $A + B$ is not defined
- if $A = \begin{bmatrix} 1 & 2.1 & 3.1 \\ 2.1 & 1.2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1.2 & 4.1 & 4 \\ 2.5 & 5 & 7 \end{bmatrix}$

\[
A + B = \begin{bmatrix} 1 + 1.2 & 2.1 + 4.1 & 3.1 + 4 \\ 2.1 + 2.5 & 1.2 + 5 & 5 + 7 \end{bmatrix} = \begin{bmatrix} 2.2 & 6.2 & 7.1 \\ 4.6 & 6.2 & 12 \end{bmatrix}
\]

$A - B$ is $A + (-1)B$
Properties

Let $A, B, C$ be $m \times n$ matrices, and $r, s \in \mathbb{R}^n$.

(P1) Self inverse of transposition: $(A^t)^t = A$

(P2) Commutativity of $\cdot$: $rA = Ar$

(P3) Associativity of $\cdot$: $r(sA) = (rs)A$

(P4) $(rA)^t = r(A^t)$

(P5) Commutativity of $+$: $A + B = B + A$

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(P9) Distributivity of transposition over $+$: $(A + B)^t = A^t + B^t$

(P10) 0 is the identity for $+$: $A + 0 = 0 + A = A$

Here 0 is $0_{m\times n}$
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Matrix times a vector

Let $A$ be an $m \times n$ matrix and $\vec{u} \in \mathbb{R}^n$.

The multiplication of $A$ by $\vec{u}$, noted $A\vec{u}$, is the $m$-dimensional vector which $i$-th component $(A\vec{u})[i]$ is the dot product of the $i$-th row vector $A[i,:]$ of $A$ and $\vec{u}$:

$$A\vec{u} = \langle A[1,:] \cdot \vec{u}, A[2,:] \cdot \vec{u}, \ldots, A[m,:] \cdot \vec{u} \rangle$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \vec{u} = \langle 1, 2, 3 \rangle$$

$$A\vec{u} = \langle 1 \times 1 + 2 \times 2 + 3 \times 3, 2 \times 1 + 3 \times 2 + 4 \times 3 \rangle = \langle 14, 20 \rangle$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{u} \in \mathbb{R}^3$$

$$I_3 \vec{u} = \vec{u}$$
Matrix times a vector

Let $A$ be an $m \times n$ matrix and $\mathbf{u} \in \mathbb{R}^n$.

The multiplication of $A$ by $\mathbf{u}$, noted $A\mathbf{u}$, is the $m$-dimensional vector which $i$-th component $(A\mathbf{u})[i]$ is the dot product of the $i$-th row vector $A[i,:]$ of $A$ and $\mathbf{u}$:

$$A\mathbf{u} = \langle A[1,:] \cdot \mathbf{u}, A[2,:] \cdot \mathbf{u}, \ldots, A[m,:] \cdot \mathbf{u} \rangle$$

Another definition of $A\mathbf{u}$ is

$$\forall 1 \leq i \leq m, (A\mathbf{u})[i] = \sum_{j=1}^{n} A[i,j] \mathbf{u}[j]$$

$A\mathbf{u}$ is defined only if the dim of $\mathbf{u}$ is the num. of cols. of $A$. 
Matrix times a vector (II)

Often we consider that $\vec{u}$ is an $n$-dim *column* vector and $A\vec{u}$ an $m$-dim *column* vector:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } \vec{u} = \langle 1, 2, 3 \rangle$$

$$A\vec{u} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 2 + 3 \times 3 \\ 2 \times 1 + 3 \times 2 + 4 \times 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \end{bmatrix}$$
Matrix times a vector (II)

Often we consider that \( \vec{u} \) is an \( n \)-dim \textit{column} vector and \( A\vec{u} \) an \( m \)-dim \textit{column} vector:

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4
\end{bmatrix}
\text{ and } \vec{u} = \langle 1, 2, 3 \rangle
\]

\[
A\vec{u} = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
= \begin{bmatrix}
1 \times 1 + 2 \times 2 + 3 \times 3 \\
2 \times 1 + 3 \times 2 + 4 \times 3
\end{bmatrix}
= \begin{bmatrix}
14 \\
20
\end{bmatrix}
\]

Remark: See \( A \) as \( n \) \( m \)-dim column vectors; then \( A\vec{u} = \sum_{j=1}^{n} \vec{u}[j]A[:,j] \)

\[
A\vec{u} = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
= 1 \begin{bmatrix}
1 \\
2
\end{bmatrix} + 2 \begin{bmatrix}
2 \\
3
\end{bmatrix} + 3 \begin{bmatrix}
3 \\
4
\end{bmatrix}
= \begin{bmatrix}
14 \\
20
\end{bmatrix}
\]
Properties of Matrix times a vector

Let $A, B$ be $m \times n$ matrices, $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

(P11) Distributivity: $(A + B)\vec{u} = A\vec{u} + B\vec{u}$, and $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

(P12) Associativity: $(rA)\vec{u} = r(A\vec{u}) = A(r\vec{u})$

(P13) $0_{m \times n}\vec{u} = \vec{0}$

(P14) $I_n\vec{u} = \vec{u}$

(P15) $A\vec{0}_n = \vec{0}_m$
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Questions: What is $A\vec{1}_n$?
What is $A\vec{e}_i$, when $i \leq n$?
Course 3 - Matrices and linear maps

I Definitions and examples

II Operations and properties
   1 Transpose
   2 Scalar multiplication
   3 Sum, difference
   4 Matrix times a vector
   5 Matrix times a matrix

III Linear maps

IV Matrices in Matlab
Matrix times a matrix

Let $A$ be a $m \times n$ matrix and $B$ be a $n \times p$ matrix. The matrix product $AB$, is the $m \times p$ matrix defined by

$$\forall 1 \leq i \leq m, \forall 1 \leq j \leq p, (AB)[i,j] = A[i,:] \bullet B[:,j]$$

$i,j$-th element is the dot product of the $i$-th line with $j$-th column

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \end{bmatrix}$ is a $3 \times 4$ matrix

$$AB = \begin{bmatrix} 1 \times 7 + 2 \times 11 \end{bmatrix}$$
Matrix times a matrix

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$$AB = \begin{bmatrix} 1 \times 7 + 2 \times 11 & 1 \times 8 + 2 \times 12 \end{bmatrix}$$
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$$= \begin{bmatrix} 29 & 32 & 35 & 38 \\ 65 & 72 & 79 & 86 \\ 101 & 112 & 123 & 134 \end{bmatrix}$$
Matrix times a matrix

Let $A$ be a $m \times n$ matrix and $B$ be a $n \times p$ matrix. The matrix product $AB$, is the $m \times p$ matrix defined by

$$\forall 1 \leq i \leq m, \forall 1 \leq j \leq p, (AB)[i,j] = A[i,:] \bullet B[:,j]$$

The algebraic definition of $AB$ is

$$\forall 1 \leq i \leq m, \forall 1 \leq j \leq p, (AB)[i,j] = \sum_{k=1}^{n} A[i,k]B[k,j]$$

$AB$ is defined only if the num. of cols of $A$ equals the num. of rows of $B$. 
Properties of matrix times a matrix

Let $A, B$ be $m \times n$ mat., $C, D$ be $n \times p$ mat., $E$ be a $p \times q$ mat., and $r \in \mathbb{R}$.

(P16) **associativity:** $A(CE) = (AC)E$

(P17) **right distributivity:** $A(C + D) = AC + AD$

(P18) **left distributivity:** $(A + B)C = AC + BC$

(P19) $r(AB) = (rA)B = A(rB)$

(P20) **right identity:** $AI_n = A$

(P21) **left identity:** $I_nC = C$

(P22) $(AB)^t = B^tA^t$

The matrix product is not commutative, even for square matrices! *i.e. in general, $A \times B \neq B \times A$*
Properties of matrix times a matrix

Let $A, B$ be $m \times n$ mat., $C, D$ be $n \times p$ mat., $E$ be a $p \times q$ mat., and $r \in \mathbb{R}$.

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(P22) $(AB)^t = B^t A^t$

**The matrix product is not commutative, even for square matrices!**

*i.e. in general,* $A \times B \neq B \times A$

**Exercise:** Prove these properties
Powers of a square matrix

Let $A$ be a square $m \times m$ matrix.

Then

$$A^2 = AA$$

and

$$A^l = A \ldots A$$

$l$ times
Example: adjacency matrix of a graph


Example: $(G^2)[1, 2] = 2$

Does this number mean something?

\[
G = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Example: adjacency matrix of a graph


Example: $(G^2)[1, 2] = 2$

Does this number mean something? nb of paths of length 2 from $v_i$ to $v_j$!

$$G = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$
Example: adjacency matrix of a graph


Example: $(G^2)[1, 2] = 2$

Does this number mean something? nb of paths of length 2 from $v_i$ to $v_j$!

What is $G^l[i,j]$?

\[
G = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
Example: adjacency matrix of a graph


Example: $(G^2)[1, 2] = 2$

Does this number mean something? nb of paths of length 2 from $v_i$ to $v_j$!

What is $G^l[i, j]$? nb of paths of length $l$ from $v_i$ to $v_j$!

$$G = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$
Course 3 - Matrices and linear maps

I Definitions and examples

II Operations and properties

III Linear maps
   1 Definition
   2 Example: geometric transformations of the plane
   3 Characterization
   4 Composition

IV Matrices in Matlab
### Definition (Linear map, or linear transformation)

A map $F : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if for any $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $a \in \mathbb{R}$:

1. $F(\vec{u} + \vec{v}) = F(\vec{u}) + F(\vec{v})$
2. $F(a\vec{u}) = aF(\vec{u})$.

In particular, $F(\vec{0}) = \vec{0}$.

**Exemples:** Let $A$ be the $2 \times 3$ matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$.

Then $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined as $F(\vec{u}) = A\vec{u}$ is linear.

Let $B$ be an $m \times n$ matrix.

Then $G : \mathbb{R}^n \to \mathbb{R}^m$ defined as $G(\vec{u}) = B\vec{u}$ is linear.
Course 3 - Matrices and linear maps

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IV Matrices in Matlab
Geometric transformations of the plane

We consider the plane \( \mathbb{R}^2 \).

**Dilation by a factor** \( a \in \mathbb{R} \): \( D_a : \mathbb{R}^2 \to \mathbb{R}^2, D_a(\vec{u}) = a\vec{u} \)

\( D_a \) is linear.

\( D_a \) is defined by \( D_a(\vec{u}) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \vec{u} = aI_2\vec{u} \).
Geometric transformations of the plane

We consider the plane $\mathbb{R}^2$.

**Dilation by a factor $a \in \mathbb{R}$:** $D_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $D_a(\vec{u}) = a\vec{u}$

$D_a$ is linear.

$D_a$ is defined by $D_a(\vec{u}) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \vec{u} = aI_2 \vec{u}$.

**Translation by a vector $\vec{v} \in \mathbb{R}^2$:** $T_{\vec{v}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T_{\vec{v}}(\vec{u}) = \vec{u} + \vec{v}$

$T_{\vec{v}}$ is not linear, because $T_{\vec{v}}(\vec{0}) = \vec{v} \neq \vec{0}$. 
Geometric transformations of the plane

We consider the plane $\mathbb{R}^2$.

**Dilation by a factor $a \in \mathbb{R}$:** $D_a : \mathbb{R}^2 \to \mathbb{R}^2$, $D_a(\vec{u}) = a\vec{u}$

$D_a$ is linear.

$D_a$ is defined by $D_a(\vec{u}) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \vec{u} = a\vec{l}_2\vec{u}$.

**Translation by a vector $\vec{v} \in \mathbb{R}^2$:** $T_{\vec{v}} : \mathbb{R}^2 \to \mathbb{R}^2$, $T_{\vec{v}}(\vec{u}) = \vec{u} + \vec{v}$

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**Counter-clockwise rotation of angle $\theta$ around $\vec{0}$:** $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$; Is $R_\theta$ linear?
Geometric transformations of the plane (II)

Counter-clockwise rotation of angle $\theta$ around $\vec{0}$: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$; Is $R_\theta$ linear?

$\vec{u} \in \mathbb{R}^2 : \exists \phi \text{ s.t. } \vec{u} = \|\vec{u}\| \langle \cos(\phi), \sin(\phi) \rangle = \langle \|\vec{u}\| \cos(\phi), \|\vec{u}\| \sin(\phi) \rangle$ then

$$R_\theta(\vec{u}) = \langle \|\vec{u}\| \cos(\phi + \theta), \|\vec{u}\| \sin(\phi + \theta) \rangle$$
Geometric transformations of the plane (II)

Counter-clockwise rotation of angle $\theta$ around $\vec{0}$: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$; Is $R_\theta$ linear?

$\vec{u} \in \mathbb{R}^2 : \exists \phi \text{ s.t. } \vec{u} = \|\vec{u}\| \langle \cos(\phi), \sin(\phi) \rangle = \langle \|\vec{u}\| \cos(\phi), \|\vec{u}\| \sin(\phi) \rangle \text{ then}$

$$R_\theta(\vec{u}) = \langle \|\vec{u}\| \cos(\phi + \theta), \|\vec{u}\| \sin(\phi + \theta) \rangle$$

$$R_\theta(\vec{u}) = \langle \vec{u}[1] \cos(\theta) - \vec{u}[2] \sin(\theta), \vec{u}[2] \cos(\theta) + \vec{u}[1] \sin(\theta) \rangle$$

Use the trigonometric identities:

- $\cos(\phi + \theta) = \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)$,
- $\sin(\phi + \theta) = \sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta)$. 
Counter-clockwise rotation of angle $\theta$ around $\vec{0}$: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$; Is $R_\theta$ linear?

$\vec{u} \in \mathbb{R}^2 : \exists \phi \text{ s.t. } \vec{u} = \|\vec{u}\|\langle \cos(\phi), \sin(\phi) \rangle = \langle \|\vec{u}\| \cos(\phi), \|\vec{u}\| \sin(\phi) \rangle$ then

$$R_\theta(\vec{u}) = \langle \|\vec{u}\| \cos(\phi + \theta), \|\vec{u}\| \sin(\phi + \theta) \rangle$$

$$R_\theta(\vec{u}) = \langle \vec{u}[1] \cos(\theta) - \vec{u}[2] \sin(\theta), \vec{u}[2] \cos(\theta) + \vec{u}[1] \sin(\theta) \rangle$$

$R_\theta$ is linear, defined by the matrix above

$$R_\theta(\vec{u}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{u}$$
Course 3 - Matrices and linear maps

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Remark: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $\vec{u} \in \mathbb{R}^n$. $F(\vec{u})$ is a $m$-dim vector that can be written:

$$F(\vec{u}) = \langle f_1(\vec{u}), \ldots, f_i(\vec{u}), \ldots, f_m(\vec{u}) \rangle$$

where the $f_i : \mathbb{R}^n \to \mathbb{R}$ are linear functions.
Characterization of linear maps

Remark: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $\vec{u} \in \mathbb{R}^n$. $F(\vec{u})$ is a $m$-dim vector that can be written:

$$F(\vec{u}) = \langle f_1(\vec{u}), \ldots, f_i(\vec{u}), \ldots, f_m(\vec{u}) \rangle = \begin{bmatrix} f_1(\vec{u}) \\ \vdots \\ f_i(\vec{u}) \\ \vdots \\ f_m(\vec{u}) \end{bmatrix}$$

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Characterization of linear maps

Remark: Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map and $\vec{u} \in \mathbb{R}^n$. 
$F(\vec{u})$ is a $m$-dim vector that can be written:

\[
F(\vec{u}) = \langle f_1(\vec{u}), \ldots, f_i(\vec{u}), \ldots, f_m(\vec{u}) \rangle = \\
\begin{bmatrix}
  f_1(\vec{u}) \\
  \vdots \\
  f_i(\vec{u}) \\
  \vdots \\
  f_m(\vec{u})
\end{bmatrix} = \\
\begin{bmatrix}
  \vec{v}_1 \cdot \vec{u} \\
  \vdots \\
  \vec{v}_i \cdot \vec{u} \\
  \vdots \\
  \vec{v}_m \cdot \vec{u}
\end{bmatrix}
\]

where the $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are linear functions.

As a consequence, it exists $m$ $n$-dim vectors $\vec{v}_1, \ldots, \vec{v}_m$ s.t. $\forall \vec{u}$:

\[
F(\vec{u}) = \langle \vec{v}_1 \cdot \vec{u}, \ldots, \vec{v}_i \cdot \vec{u}, \ldots, \vec{v}_m \cdot \vec{u} \rangle
\]
Characterization of linear maps

Remark: Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be a linear map and \( \vec{u} \in \mathbb{R}^n \).

\( F(\vec{u}) \) is a \( m \)-dim vector that can be written:

\[
F(\vec{u}) = \langle f_1(\vec{u}), \ldots, f_i(\vec{u}), \ldots, f_m(\vec{u}) \rangle = \begin{bmatrix} f_1(\vec{u}) \\ \vdots \\ f_i(\vec{u}) \\ \vdots \\ f_m(\vec{u}) \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{u} \\ \vdots \\ \vec{v}_i \cdot \vec{u} \\ \vdots \\ \vec{v}_m \cdot \vec{u} \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_i \\ \vdots \\ \vec{v}_m \end{bmatrix} \vec{u}
\]

where the \( f_i : \mathbb{R}^n \to \mathbb{R} \) are linear functions.

As a consequence, it exists \( m \) \( n \)-dim vectors \( \vec{v}_1, \ldots, \vec{v}_m \) s.t. \( \forall \vec{u} \):

\[
F(\vec{u}) = \langle \vec{v}_1 \cdot \vec{u}, \ldots, \vec{v}_i \cdot \vec{u}, \ldots, \vec{v}_m \cdot \vec{u} \rangle
\]

Proposition

Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be a linear map. Then there exists a unique \( m \times n \) matrix \( A \) such that \( \forall \vec{u} \in \mathbb{R}^n, F(\vec{u}) = A\vec{u} \).
Example: smoothing a 1D signal

Problem: Consider a *noisy* sampled signal $\tilde{u}$ of length $n$. One wants to reduce noise in this signal.

$\tilde{v}$: non-noisy signal: $n$-dimensional vector (in blue below)

$\tilde{u}$: noisy signal: $n$-dimensional vector (in red below)
Example: smoothing a 1D signal

Problem: Consider a \textit{noisy} sampled signal $\vec{u}$ of length $n$.
One wants to reduce noise in this signal.

$\vec{v}$: non-noisy signal: $n$-dimensional vector (in blue below)
$\vec{u}$: noisy signal: $n$-dimensional vector (in red below)  \textbf{De-noising operator:} from $\vec{u}$
try to recover $\vec{v}$ (or $\vec{v}' \sim \vec{v}$)
Example: smoothing a 1D signal

Problem: Consider a noisy sampled signal $\vec{u}$ of length $n$. One wants to reduce noise in this signal.

$\vec{v}$: non-noisy signal: $n$-dimensional vector (in blue below)
$\vec{u}$: noisy signal: $n$-dimensional vector (in red below)  \[ \text{De-noising operator: from } \vec{u} \text{ try to recover } \vec{v} \text{ (or } \vec{v}' \sim \vec{v}) \]

Idea: each $\vec{v}[i]$ is taken as the average of $\vec{u}[i - 1], \vec{u}[i], \vec{u}[i + 1]$

\[ f(\vec{u}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \vdots & \ddots & \vdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \vec{u}[1] \\ \vec{u}[2] \\ \vec{u}[3] \\ \vdots \\ \vec{u}[n-1] \\ \vec{u}[n] \end{bmatrix} \]

\[ f \text{ is a linear map, } f(\vec{u}) \text{ can be written as:} \]

\[ f(\vec{u}) = \begin{bmatrix} f(\vec{u})[1] \\ f(\vec{u})[2] \\ \vdots \\ f(\vec{u})[n-1] \\ f(\vec{u})[n] \end{bmatrix} \]

Graphs showing the original and smoothed signals.
Example: smoothing a 1D signal

Problem: Consider a noisy sampled signal $\bar{u}$ of length $n$. One wants to reduce noise in this signal.

De-noising operator $f$: $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$f(\bar{u})[i] = \begin{cases} \bar{u}[i] & \text{if } i = 1 \text{ or } n \\ \frac{1}{3} \bar{u}[i - 1] + \frac{1}{3} \bar{u}[i] + \frac{1}{3} \bar{u}[i + 1] & \text{otherwise} \end{cases}$$
Example: smoothing a 1D signal

Problem: Consider a *noisy* sampled signal $\vec{u}$ of length $n$. One wants to reduce noise in this signal.

De-noising operator $f$: $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$
\begin{align*}
\{ & f(\vec{u})[i] = \vec{u}[i] & \text{if } i = 1 \text{ or } n \\
& f(\vec{u})[i] = \frac{1}{3} \vec{u}[i - 1] + \frac{1}{3} \vec{u}[i] + \frac{1}{3} \vec{u}[i + 1] & \text{otherwise}
\}
\end{align*}
$$

$f$ is a linear map, $f(\vec{u})$ can be written as:

$$
f(\vec{u}) = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1/3 & 1/3 & 1/3 & \ldots & 0 \\
0 & 1/3 & 1/3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1/3 & 1/3 & 1/3 \\
0 & \ldots & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\vec{u}[1] \\
\vec{u}[2] \\
\vec{u}[3] \\
\vdots \\
\vec{u}[n-1] \\
\vec{u}[n]
\end{bmatrix}
$$
Example: smoothing a 1D signal

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\[
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\end{align*}
\]
Course 3 - Matrices and linear maps

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Definition (Composition of linear maps)

Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) and \( G : \mathbb{R}^m \to \mathbb{R}^p \) be two maps.

The \textit{composition} of \( F \) with \( G \), noted \( G \circ F \), is a map from \( \mathbb{R}^n \) to \( \mathbb{R}^p \) defined by

\[
\forall \vec{u} \in \mathbb{R}^n, G \circ F(\vec{u}) = G(F(\vec{u}))
\]
Linear maps composition

Definition (Composition of linear maps)

Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( G : \mathbb{R}^m \rightarrow \mathbb{R}^p \) be two maps. The *composition* of \( F \) with \( G \), noted \( G \circ F \), is a map from \( \mathbb{R}^n \) to \( \mathbb{R}^p \) defined by

\[
\forall \vec{u} \in \mathbb{R}^n, \; G \circ F(\vec{u}) = G(F(\vec{u}))
\]

Proposition

Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( G : \mathbb{R}^m \rightarrow \mathbb{R}^p \) be two linear maps, defined as \( F(\vec{u}) = A\vec{u} \) and \( G(\vec{v}) = B\vec{v} \).

Then \( G \circ F : \mathbb{R}^n \rightarrow \mathbb{R}^p \) is linear and is defined by

\[
\forall \vec{u} \in \mathbb{R}^n, \; G \circ F(\vec{u}) = BA\vec{u}
\]

**Proof:** straightforward!
Geometric transformations of the plane

We consider the plane $\mathbb{R}^2$.

**Dilation by a factor $a \in \mathbb{R}$:** $D_a : \mathbb{R}^2 \to \mathbb{R}^2$, $D_a(\vec{u}) = a\vec{u}$

$D_a$ is linear.

$D_a$ is defined by $D_a(\vec{u}) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \vec{u} = aI_2 \vec{u}$.

**Rotation of angle $\theta$ around $\vec{0}$:** $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, $R_\theta(\vec{u}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{u}$

**Composition of $D_a$ by $R_\theta$:** $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{u}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \vec{u}$
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Geometric transformations of the plane

We consider the plane $\mathbb{R}^2$.

Dilation by a factor $a \in \mathbb{R}$: $D_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $D_a(\vec{u}) = a\vec{u}$

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Composition of $R_\theta$ by $D_a$: $S' : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $S'(\vec{u}) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{u} = \begin{bmatrix} a \times \cos(\theta) & -a \times \sin(\theta) \\ a \times \sin(\theta) & a \times \cos(\theta) \end{bmatrix} \vec{u}$
Course 3 - Matrices and linear maps

I Definitions and examples
II Operations and properties
III Linear maps
IV Matrices in Matlab
Matrices in Matlab

Matrices are 2 dimensional arrays.

```
>> A=[1,2,3;4,5,6]
>> B=[1,2;3,4;5,6]
```

Create special matrices:

```
>> Z=zeros(2)
>> Z=zeros(2,3)
>> E=eye(2)
>> E=eye(2,3)
>> O=ones(2)
>> O=ones(2,3)
```

Arithmetic:

```
>> A+B
>> A+Z
>> A-B
>> A-Z
>> A' %transpose
>> A'- B
>> a=[5,6,7];
>> A*a %mat-vec mult.: will not work
>> A*a' %mat-vec mult.: will work
>> A*A %mat-mat mult.: will not work
>> A*B %mat-mat mult.: will work
>> A*A'
>> A.*A %elementwise multiplication
>> 2*A
>> (A*A')^2 %powering a square matrix
```
Example: smoothing a 1D signal

Problem: Consider a noisy sampled signal $\vec{u}$ of length $n$. One wants to reduce noise in this signal.

De-noising operator $f$: estimate the true value of a component as the average of the nearby components.

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$
\begin{align*}
  f(\vec{u})[i] &= \vec{u}[i] & \text{if } i = 1 \text{ or } n \\
  f(\vec{u})[i] &= \frac{1}{3} \vec{u}[i - 1] + \frac{1}{3} \vec{u}[i] + \frac{1}{3} \vec{u}[i + 1] & \text{otherwise}
\end{align*}
$$
Example: smoothing a 1D signal

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De-noising operator $f$: estimate the true value of a component as the average of the nearby components.

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$f(\vec{u})[i] = \begin{cases} \vec{u}[i] & \text{if } i = 1 \text{ or } n \\ \frac{1}{3} \vec{u}[i - 1] + \frac{1}{3} \vec{u}[i] + \frac{1}{3} \vec{u}[i + 1] & \text{otherwise} \end{cases}$$

$f$ is a linear map, $f(\vec{u})$ can be written as:

$$f(\vec{u}) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \ldots & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \ldots & 0 \\ \vdots \\ 0 & \ldots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{u}[1] \\ \vec{u}[2] \\ \vec{u}[3] \\ \vdots \\ \vec{u}[n - 1] \\ \vec{u}[n] \end{bmatrix}$$
Example: smoothing a 1D signal

Problem: Consider a noisy sampled signal $\tilde{u}$ of length $n$. One wants to reduce noise in this signal.

De-noising operator $f$: $f$ is a linear map, $f(\tilde{u})$ can be written as:

$$
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1/3 & 1/3 & 1/3 & \ldots & 0 \\
0 & 1/3 & 1/3 & \ldots & 0 \\
\vdots & & & & \\
0 & \ldots & 1/3 & 1/3 & 1/3 \\
0 & \ldots & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\tilde{u}[1] \\
\tilde{u}[2] \\
\tilde{u}[3] \\
\vdots \\
\tilde{u}[n-1] \\
\tilde{u}[n] \\
\end{bmatrix}
$$

Naive implementation:

```matlab
A=eye(n);
for i=2:n-1
    A(i,i-1:i+1) = [1/3, 1/3, 1/3];
end
A*u;
```
Example: smoothing a 1D signal

Problem: Consider a *noisy* sampled signal $\vec{u}$ of length $n$.
One wants to reduce noise in this signal.

De-noising operator $f$: $f$ is a linear map, $f(\vec{u})$ can be written as:

$$f(\vec{u}) = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1/3 & 1/3 & 1/3 & \ldots & 0 \\
0 & 1/3 & 1/3 & \ldots & 0 \\
\vdots & & & & \\
0 & \ldots & 1/3 & 1/3 & 1/3 \\
0 & \ldots & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\vec{u}[1] \\
\vec{u}[2] \\
\vec{u}[3] \\
\vec{u}[n-1] \\
\vec{u}[n]
\end{bmatrix}.$$

Naive implementation: the signal is sampled at 44100Hz, is 4s long

```matlab
A=eye(n);
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    A(i,i-1:i+1) = [1/3, 1/3, 1/3];
end
A*u;
```
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1 & 0 & 0 & \ldots & 0 \\
1/3 & 1/3 & 1/3 & \ldots & 0 \\
0 & 1/3 & 1/3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1/3 & 1/3 & 1/3 \\
0 & \ldots & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix} \vec{u}[1] \\ \vec{u}[2] \\ \vec{u}[3] \\ \vdots \\ \vec{u}[n-1] \\ \vec{u}[n] \end{bmatrix}$$

Naive implementation: the signal is sampled at 44100Hz, is 4s long
$\Rightarrow$ use sparse matrices
Sparse matrices

See https://www.mathworks.com/help/matlab/ref/sparse.html

Example: to create

\[
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1/3 & 1 & 0 & \ldots & 0 \\
0 & 1/3 & 1 & \ldots & 0 \\
\vdots \\
0 & \ldots & 1/3 & 1 & 0 \\
0 & \ldots & 0 & 0 & 1
\end{bmatrix}
\]

do

\[
A = \text{sparse}(1:n,1:n,1,n,n) + \text{sparse}(2:n-1,1:n-2,1/3,n,n);
\]