Exercise 5.1. Consider the function
\[ f(x) = \cos(x) + \frac{1}{2}\sin(x) \]
and suppose that we wish to approximate \( f \) in the interval \([-1, 3]\) (where the units measure radians).

For \( n = 1, n = 2, \text{and} n = 3 \):

(a) Compute the coefficients of the interpolating polynomial \( p_n(x) \) of degree at most \( n \) that matches the values of \( f(x) \) at \( n + 1 \) equally spaced points in \([-1, 3]\).

(b) Plot the (superimposed) values of \( f(x) \) and \( p_n(x) \) at 30 equally spaced points in \([-1, 3]\), also designating the \( n + 1 \) interpolation points in your plots.

(c) Comment on whether, in your opinion, each \( p_n \) gives a good fit to \( f(x) \) in this interval.

Exercise 5.2. Consider the interval \([-1, 1]\) and Runge’s function:
\[ \rho(x) = \frac{1}{1 + 25x^2}. \] (5.1)

(a) For \( n = 9 \) and \( n = 16 \):

(i) Compute the interpolating polynomial \( p_n(x) \) of degree at most \( n \) that matches \( \rho(x) \) at \( n + 1 \) equally spaced points in \([-1, 1]\) (including the endpoints).

(ii) Print the interpolation points and the coefficients of \( p_n(x) \) expressed in monomial form.

(iii) Evaluate \( p_n(x) \) at 51 equally spaced points in the interval \([-1, 1]\), compute the error
\[ e_n(x) = \rho(x) - p_n(x) \]
at each of these points, and plot these errors. Comment on your results, in particular on any differences in the size of the error \( e_n(x) \) in different regions of the interval.

(b) For \( n = 9 \) and \( n = 16 \):

(i) Compute the interpolating polynomial \( r_n(x) \) of degree at most \( n \) that matches \( \rho(x) \) at the following \( n + 1 \) points:
\[ z_j = \cos\left(\frac{(j + \frac{1}{2})\pi}{n + 1}\right), \quad j = 0, 1, \ldots, n. \] (5.2)

These are the zeros of the Chebyshev polynomial \( T_{n+1}(x) \). For example, when \( n = 2 \), these points are the zeros of \( T_3(x) = 4x^3 - 3x \), namely \( z_0 = \cos(\pi/6) = \sqrt{3}/2 \approx 0.86603 \), \( z_1 = \cos(\pi/2) = 0 \), and \( z_2 = \cos(5\pi/6) \approx -0.86603 \).

(ii) Print the interpolation points \( \{z_k\}, k = 0, \ldots, n, \) from (5.2) and the coefficients of the polynomial \( r_n(x) \), expressed in monomial form.

(iii) Evaluate the polynomial \( r_n(x) \) at 51 equally spaced points in the interval \([-1, 1]\), and compute the error \( \tilde{e}_n(x) = \rho(x) - r_n(x) \) at those points. Plot \( \tilde{e}_n(x) \) and comment on your results.
(c) Comment on any notable differences between the results of (a) and (b) in terms of the accuracy of the interpolating polynomials \( p_n(x) \) and \( r_n(x) \).

**Exercise 5.3.** Consider the following table of 5 points \( x_0, \ldots, x_4 \) and corresponding values of two different functions, \( f \) and \( g \), evaluated at \( \{x_i\} \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f_i )</th>
<th>( g_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2.9</td>
<td>2.9</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>2.3</td>
<td>3.8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>2.0</td>
<td>4.1</td>
</tr>
</tbody>
</table>

(a) Interpolate the values of \( \{f_i\} \) at \( \{x_i\} \) using
   (i) the interpolating polynomial of degree at most 4 (using, for example, Matlab’s `polyfit` and `polyval` commands);
   (ii) the ‘not a knot’ default spline computed by Matlab’s `spline` command;
   (iii) the ‘clamped’ spline computed by Matlab in which first derivatives at the endpoints of the interval are forced to be zero, as illustrated in the example labeled ‘Spline Interpolation of Distribution and Specify Endpoint Slopes’ from Matlab’s ‘help’ about `spline`;
   (iv) the shape-preserving piecewise cubic computed by Matlab’s `pchip` command.

For each of these cases, compute the values of the interpolants at a reasonable number (say, 41) of equally spaced points in the interval \([-1, 7]\). Plot the computed interpolants, both separately and superimposed.

Comment on how the interpolants differ qualitatively. In your opinion, which choices seem to provide the best and worst “feeling” for the data? Explain why.

(b) Repeat part (a), this time interpolating the values of \( \{g_i\} \) at \( \{x_i\} \). Be sure to comment on qualitative differences (if any) between the interpolants, and on any qualitative differences from the results of part (a).

(c) Make up your own set of function values to be associated with the given 5 points \( \{x_i\} \), choosing the function values so that “interesting” things happen with one or more of the interpolants. Comment on how you chose the function values, and how the various interpolants differ qualitatively for your data. Which choice of interpolant seems to you to provide the best feeling for your data? Explain why.

**Exercise 5.4.** This problem involves determination of a linear polynomial \( ax + b \) that approximates the function \( f(x) = x^3 \) in the interval \([-1, 1]\), using two different norms.

(a) First, we seek an approximation that minimizes the squared \( \ell_2 \) norm of \( f \) in \([-1, 1]\):

\[
E_2^2(f, a, b) = \int_{-1}^{1} (x^3 - (ax + b))^2 \, dx.
\]

(i) Find the values of \( a \) and \( b \), denoted by \( a_2 \) and \( b_2 \), for which \( E_2^2(f, a, b) \) is minimized. Show and explain how you obtained your answers.

(ii) What is the value of \( E_2^2(f, a_2, b_2) \)?

(iii) For a reasonable number of equally spaced points in \([-1, 1]\), make two plots: one showing both \( x^3 \) and \( a_2x + b_2 \), and the other showing the error \( x^3 - a_2x - b_2 \) at each point. Where in the interval does the maximum error occur?
(b) Now consider the linear function that gives the best fit to \( f(x) = x^3 \) in the \textit{infinity norm}, i.e., that minimizes
\[
E_\infty(f, a, b) = \|x^3 - (ax + b)\|_\infty = \max_{x \in [-1, 1]} |x^3 - (ax + b)|.
\]

(i) Find the values of \( a \) and \( b \), denoted by \( a_\infty \) and \( b_\infty \), for which \( E_\infty(f, a, b) \) is minimized. Show and explain how you obtained your answers.

(ii) What is the value of \( E_\infty(f, a_\infty, b_\infty) \)?

(iii) For a reasonable number of equally spaced points in \([-1, 1]\), make a plot showing both \( x^3 \) and \( ax_\infty + b_\infty \), and a plot showing the error \( x^3 - ax_\infty - b_\infty \) at each point. Where in the interval does the maximum-magnitude error occur? Does the behavior of the error match the relevant theory?

Exercise 5.5. Assume that you wish to approximate a nonlinear function \( f(t) \) evaluated at a set of points \( \{t_i\}, i = 1, \ldots, m \) with a cubic polynomial \( p_3(t) \) by minimizing the two-norm of the vector whose \( i \)th component is \( f(t_i) - p_3(t_i) \).

(i) Formulate the problem of determining the coefficients in the polynomial as a linear least-squares problem \( \min \|b - Ax\|_2^2 \), i.e., define the components of the unknown vector \( x \), and express \( A \) and \( b \) in terms of \( t_i \) and \( f(t_i) \).

(ii) Solve the least-squares problem given in (i) when \( f(t) = e^t, m = 11 \), and \( t_i = (i - 1)/100 \). Give the optimal solution, the residual and the two-norm of the residual. What is \( \text{cond}(A) \)? Do the coefficients appear to be related to the Taylor-series expansion of the exponential around \( t = 0 \)? Explain.

(iii) Perform the same calculations as in part (ii), keeping \( f(t) \) and \( m \) the same, but letting \( t_i = (i - 1)/10 \). What is \( \text{cond}(A) \) in this case? Comment on any differences from part (ii).

Exercise 5.6. Suppose that you are given the following tabulated data \( \{x_i, f_i\}, i = 0, \ldots, n \), where \( f_i = f(x_i) \) for some function \( f \):

\[
\begin{array}{ccc}
  i & x_i & f_i \\
  0 & 0 & 1.0 \\
  1 & 1 & 0.52 \\
  2 & 3 & 0.68 \\
  3 & 4 & 2.12 \\
\end{array}
\]

Assume that interval \( i \) means \([x_{i-1}, x_i]\).

(i) Consider the following piecewise cubic, where \( s_i \) is the local cubic in interval \( i \), and \( u = x - x_{i-1} \) if \( x \) is in interval \( i \):

Interval 1: \( s_1(x) = 1 - 0.5u + 0.02u^3 \)
Interval 2: \( s_2(x) = 0.52 - 0.44u + 0.06u^2 + 0.1u^3 \)
Interval 3: \( s_3(x) = 0.68 + u + 0.66u^2 - 0.22u^3 \)

Is this the interpolating \textit{natural} piecewise cubic spline for these data? Explain why or why not, and show your associated calculations.

(ii) Would your answer be different (for the same piecewise cubic) if you were instead given the tabulated data below? Explain.

\[
\begin{array}{cc}
  x_i & f_i \\
  0 & 1.0 \\
  1 & 0.4 \\
  3 & 0.68 \\
  4 & 1.1 \\
\end{array}
\]
Exercise 5.7. Given a scalar $b > 1$, let $I(b)$ denote the exact integral

$$I(b) = \int_1^b \frac{dt}{t} = \ln b.$$ 

For a given value of $b$, let $M(b)$ denote the estimate of $I(b)$ from the midpoint rule; $T(b)$ the estimate of $I(b)$ from the trapezoid rule; and $S(b)$ the estimate of $I(b)$ from Simpson’s rule. For $b = 1.5$ and $b = 2$, do the following:

(i) compute the exact integral $I(b)$;
(ii) compute $M(b)$ and the error $I(b) - M(b)$;
(iii) compute $T(b)$ and the error $I(b) - T(b)$;
(iv) compute $S(b)$ and the error $I(b) - S(b)$;
(v) comment on how well the actual errors correspond to the estimates derived in class. Why are the error estimates more accurate for the first value of $b$?