Proof of \( n = o(2^n) \)

**Proposition.** \( n = o(2^n) \).

*Proof.* By the definition of small-o, the statement of the proposition is equivalent to \( \lim_{n \to \infty} \frac{n}{2^n} = 0 \). We will prove this equality separating the exponent \( 2^n \) into two parts \( 2^n = 2^n/2 \cdot 2^n/2 \). One part will be “responsible” for neutralizing \( n \), and the other part will be “responsible” for tending to zero.

First we prove that \( 2^n/2 > n/2 \) for all positive integers \( n \geq 1 \) by induction on \( n \). Base cases for \( n = 1, 2, 3 \) can be checked by direct computation. Assume that \( 2^n/2 > n/2 \) for some \( n \geq 3 \). Then

\[
2^{(n+1)/2} > \sqrt{2} \cdot 2^n/2 > \sqrt{2} \cdot n/2 > n/2 + 0.4 \cdot n/2 > (n + 1)/2.
\]

Now we will prove that the limit \( \lim_{n \to \infty} \frac{n}{2^n} \) is equal to zero. Assume that we are given with \( \varepsilon > 0 \) and we want to find \( N \) such that \( \frac{m}{2^m} < \varepsilon \) for every \( m > N \). In order to find such \( N \), we use already proved inequality in order to produce the following bound:

\[
\frac{m}{2^m} = \frac{m/2}{2^{m/2}} \cdot \frac{2}{2^{m/2}} < \frac{2}{2^{m/2}}.
\]

Since \( \frac{2}{2^{m/2}} \) decreases when \( m \) grows, we can set \( N \) equal to the point where \( \varepsilon = \frac{2}{2^{N/2}} \). Solving this equation with respect to \( N \) (check it by yourself!), we obtain \( N = 2 \log_2(\varepsilon/2) \). Then for \( m > N \) we have

\[
\frac{m}{2^m} < \frac{2}{2^{m/2}} < \frac{2}{2^{N/2}} = \varepsilon.
\]

So, \( \lim_{n \to \infty} \frac{n}{2^n} = 0 \), and the proposition is proved. \( \square \)