Math Facts
Asymptotic notation

Motivation:
Suppose that on inputs of size $n$, Algorithm 1 takes time

$$f(n) = n^2 + 10n$$

while Algorithm 2 takes time

$$g(n) = 10n \log_2 n + 100n$$

Since

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

Eventually, as $n$ gets larger, Algorithm 1 will get slower and slower than Algorithm 2.
Input interpretation:

\[
\begin{array}{|c|c|}
\hline
\text{plot} & \frac{n^2 + 10n}{10n \log_2(n) + 100n} \quad n = 1 \text{ to } 500 \\
\hline
\end{array}
\]
In this class, we are mainly concerned with the *rate of growth* of the running time as a function of the *input size*

For this purpose, we really only need to worry about the “high order term” of this function

We typically ignore the leading constant

- For $f(n) = n^2 + 10n$, we just say $f(n) = O(n^2)$
- For $g(n) = 10n \log_2 n + 100n$, we just say $g(n) = O(n \log_2 n)$
- Since $\log_b n = \log n / \log b$, we can just say $g(n) = O(n \log n)$ — the base of the logarithm doesn’t really matter (*as long as it is a constant*).

The key idea: for large enough $n$, an $O(n \log n)$ algorithm will be faster than an $O(n^2)$ algorithm.
**Some formal definitions**

**Definition:** Let $f$ and $g$ be functions from $\mathbb{Z}_{>0}$ to $\mathbb{R}$. We say $f = O(g)$ if

$$|f(n)| \leq c|g(n)|$$

for some constant $c$ and all sufficiently large $n$

Or more precisely:

$$\exists c \in \mathbb{R}_{>0} \ \exists n_0 \in \mathbb{Z}_{>0} \ \forall n \geq n_0 : \ |f(n)| \leq c|g(n)|$$

**Intuition:** $f = O(g)$ means $f$ grows *no faster* than $g$
Example: \( g(n) = 10n \log_2 n + 100n \)

Claim: \( g(n) = O(n \log_2 n) \)

Need to find \( c \) and \( n_0 \) such that
\[
10n \log_2 n + 100n \leq cn \log_2 n
\]
for all \( n \geq n_0 \)

Let’s try \( c = 20 \) and solve for \( n_0 \):

\[
10n \log_2 n + 100n \leq 20n \log_2 n
\]
\[
100n \leq 10n \log_2 n
\]
\[
10 \leq \log_2 n
\]

So \( n_0 = 2^{10} = 1024 \) does the job
Implicit big-$O$ notation

If $f(n) = 2n^2 + 10n + 1$, we may also write

$$f(n) = 2n^2 + O(n)$$

This means that

$$f(n) = 2n^2 + h(n)$$

for some function $h = O(n)$

Useful in situations where we do not want to completely ignore the constant in the high-order term
big-$\Theta$ notation

**Definition:** Let $f$ and $g$ be functions from $\mathbb{Z}_{>0}$ to $\mathbb{R}$.

We say $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$

Equivalently:

$$|f(n)| \leq c|g(n)| \leq d|f(n)|$$

for some constants $c, d$ and all sufficiently large $n$

**Intuition:** $f = \Theta(g)$ means $f$ and $g$ grow at the same rate

**Note:** $f = \Theta(g)$ is a symmetric relation:

$$f = \Theta(g) \iff g = \Theta(f)$$
Example: \( f(n) = n^2 + 10n, \ g(n) = 2n^2 + n \log_2 n \)

We have \( f = \Theta(g) \)

Example: \( f(n) = 2n^2 + 10n, \ g(n) = n^3 + n \)

- \( f = O(g) \)
- \( g \neq O(f) \)
little-o notation

**Definition:** Let $f$ and $g$ be functions from $\mathbb{Z}_{>0}$ to $\mathbb{R}$. We say $f = o(g)$ if

$$\lim_{{n \to \infty}} \frac{f(n)}{g(n)} = 0$$

Equivalently: for every $\epsilon \in \mathbb{R}_{>0}$

$$|f(n)| < \epsilon |g(n)|$$

for all sufficiently large $n$

**Intuition:** $f = o(g)$ means $f$ grows *more slowly* than $g$
Example: $f(n) = 2n^2 + 10n$, $g(n) = n^3 + n$

Calculate the limit:

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{2n^2 + 10n}{n^3 + n} = \lim_{n \to \infty} \frac{2/n + 10/n^2}{1 + 1/n^2}
\]

\[
= \frac{0 + 0}{1 + 0} = 0
\]

\[\therefore f = o(g)\]

**General fact:** if $f = o(g)$, then

- $f = O(g)$
- $g \neq O(f)$
Limit Comparison Theorem: *general tool for comparing growth rates*

Let $f$ and $g$ be functions from $\mathbb{Z}_{>0}$ to $\mathbb{R}$.

Suppose

$$0 \leq L := \lim_{n \to \infty} \frac{|f(n)|}{|g(n)|} \leq \infty$$

Then we have:

- If $L = 0$ then $f = o(g)$
- If $L = \infty$ then $g = o(f)$
- If $0 < L < \infty$ then $f = \Theta(g)$
A more concrete tool

Suppose $f$ is a sum of terms, where each term of the form $cn^\alpha \log(n)^\beta$, for constants $c, \alpha, \beta$, where $c \neq 0$

We can sort the terms:

- $cn^\alpha \log(n)^\beta$ is a higher term than $c'n^{\alpha'} \log(n)^{\beta'}$ if
  - $\alpha > \alpha'$, or
  - $\alpha = \alpha'$ and $\beta > \beta'$

For two such functions $f$ and $g$, we can compare their highest terms:

- If $g$’s highest term is higher than $f$’s: $f = o(g)$
- If $f$’s highest term is higher than $g$’s: $g = o(f)$
- Otherwise: $f = \Theta(g)$
## Pop Quiz!

<table>
<thead>
<tr>
<th>$f(n)$</th>
<th>$g(n)$</th>
<th>$f = o(g)$?</th>
<th>$g = o(f)$?</th>
<th>$f = \Theta(g)$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3n^2$</td>
<td>$2n^3$</td>
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<tr>
<td>$2n^3 + 10n^2$</td>
<td>$n^2 + n$</td>
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<tr>
<td>$n^2 \log_2 n + 10n^2$</td>
<td>$n^3 - n$</td>
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<td>$n^2 / \log_2 n$</td>
<td>$n \log_2 n$</td>
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<td>$(\log_2 n)^2$</td>
<td>$\sqrt{n}$</td>
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<td>$\log_5 n$</td>
<td>$\log_3 n$</td>
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<tr>
<td>$5^n$</td>
<td>$3^n$</td>
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Counting steps

for $i$ in $[1..n]$ do
  for $j$ in $[1..i]$ do
    HERE

How many times does line “HERE” get executed?

For each iteration of the outer for loop, it gets executed $i$ times:

$$S = \sum_{i=1}^{n} i = n(n+1)/2 = n^2/2 + O(n) = O(n^2)$$
The tale of young Gauss and the sum of the first 100 positive integers

Old Gauss (1777–1855) on German banknote

\[
S = \frac{1 + 2 + \cdots + 99 + 100}{2S = \frac{100 + 99 + \cdots + 2 + 1}{2S = 100 \cdot 101}
\]

What about more complicated sums? \( \sum_i i^2, \sum_i \frac{1}{i} \)
Approximating sums by integrals (1)

Suppose $f$ is continuous and non-decreasing on $[a, b]$, where $a, b \in \mathbb{Z}$

\[
\sum_{i=a}^{b-1} f(i) \leq \int_a^b f(x) \, dx
\]
Approximating sums by integrals (2)

Suppose $f$ is continuous and non-decreasing on $[a, b]$, where $a, b \in \mathbb{Z}$

$$\int_{a}^{b} f(x) \, dx \leq \sum_{i=a+1}^{b} f(i)$$
Approximating sums by integrals (3)

More general statement

Suppose $f$ is continuous and monotone (non-decreasing or non-increasing) on $[a, b]$, where $a, b \in \mathbb{Z}$.

Then

$$
\int_a^b f(x)dx + m \leq \sum_{i=a}^{b} f(i) \leq \int_a^b f(x)dx + M,
$$

where $m := \min(f(a), f(b))$, and $M := \max(f(a), f(b))$.
Example: Estimate $Q_n := \sum_{i=1}^{n} i^2$

$$\int_{1}^{n} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{1}^{n} = \frac{n^3}{3} - \frac{1}{3}$$

Therefore: $n^3/3 + 2/3 \leq Q_n \leq n^3/3 + n^2 - 1/3$

$Q_n = n^3/3 + O(n^2)$

Example: Estimate $H_n := \sum_{i=1}^{n} 1/i$

$$\int_{1}^{n} (1/x) \, dx = \left[ \ln(x) \right]_{1}^{n} = \ln(n)$$

Therefore: $\ln(n) + 1/n \leq H_n \leq \ln(n) + 1$

$H_n = \ln(n) + O(1)$

$H_n$ is called the $n$th Harmonic Number
Geometric series

\[ S_n(r) := \sum_{i=0}^{n} r^i = 1 + r + r^2 + \cdots + r^n \]

**Fact:** for \( r \neq 1 \), we have

\[ S_n(r) = \frac{1 - r^{n+1}}{1 - r} \]

**Proof:**

\[(1 - r)S_n(r) = (1 + r + \cdots + r^n) - (r + r^2 + \cdots r^{n+1}) = 1 - r^{n+1} \]

**Example:** \( S_n(2) = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1 \)

**Example:** \( S_n(1/2) = 1 + 1/2 + 1/4 + \cdots + 1/2^n = 2 - 1/2^n \)
Infinite Series

$$\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=0}^{n} a_i$$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{2^i} = \lim_{n \to \infty} (2 - \frac{1}{2^n}) = 2$$

More generally, for $|r| < 1$:

$$\sum_{i=0}^{\infty} r^i = \lim_{n \to \infty} \sum_{i=0}^{n} r^i = \lim_{n \to \infty} \frac{(1 - r^{n+1})}{(1 - r)} = \frac{1}{(1 - r)}$$
Consider $\sum_{i=0}^{\infty} a_i$, where $a_i \geq 0$

**Fact:** either $\sum_{i=0}^{\infty} a_i$ converges to a constant, or diverges to $\infty$

Clearly, if $\sum_{i=0}^{\infty} a_i < \infty$, then $a_i \to 0$

But the converse is not true!

It may the the case that $\sum_{i=0}^{\infty} a_i = \infty$, even if $a_i \to 0$

**Example:**

$$\sum_{i=1}^{\infty} \frac{1}{i} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i} = \lim_{n \to \infty} \left( \ln(n) + O(1) \right) = \infty$$
The limit ratio test

Consider the series $\sum_{i=0}^{\infty} a_i$, where $a_i \geq 0$

Let $L := \lim_{i \to \infty} \frac{a_{i+1}}{a_i}$

- If $L < 1$, the series converges
- If $L > 1$, the series diverges to infinity
- If $L = 1$ (or the limit does not exist), the test is inconclusive

Example:

\[
\sum_{i=0}^{\infty} \frac{i}{2^i} : \quad \frac{a_{i+1}}{a_i} = \frac{i+1}{2i} \to \frac{1}{2} \quad \therefore \text{converges}
\]

Example:

\[
\sum_{i=1}^{\infty} \frac{1}{i} : \quad \frac{a_{i+1}}{a_i} = \frac{i}{i+1} \to 1 \quad \therefore \text{inconclusive}
\]