1. Let $\mathcal{L} = \mathcal{L}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \subseteq \mathbb{R}^n$ be a rank n lattice and let $\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n$ be the Gram-Schmidt orthogonalization of $\mathbf{b}_1, \ldots, \mathbf{b}_n$.
   (a) Show that it is *not* true in general that $\lambda_n(\mathcal{L}) \geq \max_i \|\tilde{\mathbf{b}}_i\|_2$.
   (b) Show that for any $j = 1, \ldots, n$, $\lambda_j(\mathcal{L}) \geq \min_{i=j,\ldots,n} \|\tilde{\mathbf{b}}_i\|_2$.
   (c) Show that for any $\mathbf{x} \in \mathbb{R}^n$, there exists a point $\mathbf{y} \in \mathcal{L}$ such that $\|\mathbf{x} - \mathbf{y}\|_2^2 \leq \frac{1}{4} \sum_{i=1}^n \|\tilde{\mathbf{b}}_i\|_2^2$.
   (d) Show that $C = \{\mathbf{x} \in \mathbb{R}^n : -\frac{1}{2}\|\tilde{\mathbf{b}}_i\|_2^2 \leq \langle \mathbf{x}, \tilde{\mathbf{b}}_i \rangle < \frac{1}{2}\|\tilde{\mathbf{b}}_i\|_2^2, \forall i \in [n]\}$ is a fundamental domain of $\mathcal{L}$.
   (e) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{L}$ be linearly independent vectors. Show that there exists a basis $\mathbf{y}_1, \ldots, \mathbf{y}_n$ of $\mathcal{L}$ such that $\|\tilde{\mathbf{y}}_i\|_2 \leq \|\tilde{\mathbf{v}}_i\|_2$ and $\|\mathbf{y}_i\|_2^2 \leq \|\tilde{\mathbf{v}}_i\|_2^2 + \frac{1}{4} \sum_{j=1}^{i-1} \|\tilde{\mathbf{v}}_j\|_2^2$ for all $i \in [n]$.

2. (a) For all large enough $n \in \mathbb{Z}$, find an n-dimensional full-rank lattice in which the successive minima $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (in the l_2 norm) do not form a basis of the lattice. (Hint: Cesium Chloride)
   (b) Show that for any 2-dimensional full-rank lattice $\mathcal{L}$, the successive minima $\mathbf{v}_1, \mathbf{v}_2$ *do* form a basis of $\mathcal{L}$. (Hint: consider the lattice obtained by projecting $\mathcal{L}$ on the one-dimensional subspace $\{\mathbf{v}_1\}^\perp$ and show that the projection of $\mathbf{v}_2$ must be a basis of this lattice)
   (c) Let $\mathcal{L}$ be a 2 dimensional lattice, and let $\mathbf{b}_1, \mathbf{b}_2$ be an δ-LLL reduced basis with $\delta = 1$. Show that $\|\mathbf{b}_1\|_2 = \lambda_1(\mathcal{L})$.
   (d) Among all 2-dimensional full-rank lattices with $\lambda_1(\mathcal{L}) = 1$, which one has the smallest $\det(\mathcal{L})$? (this lattice is unique up to rotation).

3. Show that a δ-LLL reduced basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of a lattice $\mathcal{L}$ with $\delta = \frac{3}{4}$ satisfies the following properties.
   (a) $\|\mathbf{b}_1\|_2 \leq 2^{(n-1)/4}(\det(\mathcal{L}))^{1/n}$.
   (b) For any $1 \leq i \leq n$, $\|\mathbf{b}_i\|_2 \leq 2^{(i-1)/2}\|\tilde{\mathbf{b}}_i\|_2$.
   (c) For any $1 \leq i \leq n$, $\lambda_i(\mathcal{L}) \geq 2^{-(n-1)/2}\|\mathbf{b}_i\|_2$.
   (d) Let $\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*$ denote the associated dual basis. Show that $\|\mathbf{b}_n^*\|_2 \leq 2^{(n-1)/2}\lambda_1(\mathcal{L}^*)$. (Hint: Use Exercise 1 from the last homework)

4. Show that the LLL analysis is essentially tight: for $1/4 < \delta \leq 1$, construct an δ-LLL reduced basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ for some lattice $\mathcal{L}$ such that $\|\mathbf{b}_1\|_2 \geq \sqrt{1 - 1/(4\delta)}(2/\sqrt{4\delta - 1})^{n-1}\lambda_1(\mathcal{L})$.