1. Consider the following methods for numerical differentiation: approximate values of the first derivative of a function $f(x)$ are given by

$$\frac{f(x + h) - f(x)}{h},$$

or

$$\frac{f(x + h) - f(x - h)}{2h}.$$

Here $h$ is chosen appropriately by the user.

(a) Which of these two methods is to be preferred? Discuss the approximation error and the error associated with roundoff.

(b) In particular, provide some guidelines on how to select $h$.

2. (a) Let $x_i, i = 0, \ldots, n$, be distinct real numbers. How do we know that there is a unique polynomial $p_n$ of degree $n$, which interpolates given values $f_i$ at the points $x_i$?

(b) Describe the Newton algorithm to determine the polynomial interpolant of a set $(x_i, f_i), i = 0, \ldots, n$.

(c) Assume that we have determined $p_n$ for such a set. Now arbitrarily permute the pairs $(x_i, f_i)$. Use the Newton algorithm again. How can we know that we get the same interpolating polynomial?
3. (a) What is the trapezoidal rule? For which family of functions is this formula exact?

(b) The error when using this and many other numerical quadrature rules can be estimated by formulas such as

\[ I_h(f) - I(f) = Ch^q(d/dx)^r f(\xi), \quad h = b - a, \quad \xi \in (a, b), \]

where \( C \) is constant, and \( q \) and \( r \) are integers. Determine these three parameters for the trapezoidal rule giving as full an explanation as you can.

(c) Let \( I_{h/2}(f) \) denote the result of using the trapezoidal rule on the interval \([a, b]\) and where \( h = b - a \). Subdivide the interval into two equal parts, use the trapezoidal rule for each of them, and add the two results to obtain \( I_h(f) \). Then, choose \( \alpha \) in the formula

\[ \alpha I_h(f) + (1 - \alpha)I_{h/2}(f) \]

so that we get the best possible result for smooth functions \( f(x) \). Can a formula such as in (b) be derived for the resulting numerical quadrature rule, and if so, what are the parameters \( C, q, \) and \( r \)?

4. Consider a function \( f(x) \), which is \( 2\pi \)-periodic, i.e.,

\[ f(x + 2\pi) = f(x), \quad \forall x. \]

(a) Describe briefly the trigonometric interpolation problem associated with such functions and an equidistant collection of interpolation points.

(b) Assume that the number of interpolation points, \( n \), satisfies \( n = 2^\ell \), where \( \ell \) is an integer. What is the expense of computing the trigonometric interpolant as a function of \( n \) if we use the appropriate, celebrated algorithm?

(c) Can cubic splines be used to approximate periodic functions, in particular, can we construct a variant of the cubic splines that are periodic? If this is possible, what are the changes required of the algorithm? Is the matrix of the resulting linear system tridiagonal? What are the cost if we use \( n \) spline nodes?
5. Let the function $f(x)$ be defined on $[-1, 1]$. A standard error estimate for the polynomial interpolation problem is,

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_{n+1}(x),$$

where

$$\omega_{n+1}(x) = (x - x_0)(x - x_1) \ldots (x - x_n).$$

(a) What are the parameters $x_i$?

(b) How large can the maximum of $|\omega_{n+1}(x)|$ be if we are free to choose the $x_i$ freely (and badly) in the interval $[-1, 1]$?

(c) What is a good choice of the $x_i$? Explain several benefits of the selection that you propose and provide as many details as you can.