Assignment Set 3, G22.2421/G63.2020
Spring 2003
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The following assignments are due in three weeks.

The idea of this exercise is to learn about adaptive step size control in the solution of ordinary differential equations. This topic is covered in Chapter 5 of Iserles’ book.

Develop a program for an extrapolated Runge-Kutta method. It is based on the standard fourth order Runge-Kutta method, which has been discussed in class. The estimate of the error, on which the step length control is based, is obtained by comparing the result of one Runge-Kutta step, using a proposed step size $h_{old}$, with the normally more accurate value obtained by taking two Runge-Kutta steps with step size $h_{old}/2$ and then using an extrapolation procedure, such as the one discussed in class for Euler’s method. The estimated error, $e$, obtained in this way is compared with a prescribed tolerance and the proposed new value, given by the extrapolated value at $t_{old} + h_{old}$, is either accepted or rejected. If it is rejected, a new step size is computed by using the formula

$$h_{new} = \min \left( q h_{old}, h_{max}, \left( \frac{\rho \ast TOL}{\|e\|} \right)^{1/5} h_{old} \right),$$

and a new attempt to advance the solution is carried out. Here $TOL$ is the given tolerance, and $q$ and $\rho$ tuning parameters chosen by the user. Here $q > 1$, and $\rho < 1$; typical values are 5 and 0.9, respectively. $h_{max}$ is selected in some reasonable way by the user.

If the step is accepted, we attempt to use the value $h_{new}$ as the time step of the next step. However, the step size should be decreased if it is so long that it would carry us past the final value of $t$ that we have in mind.
1. Use your program to solve the following problem, which describes the motion of a satellite under the gravitational influence of the Earth and its Moon:

\[ x_1'' = x_1 + 2x_2' - \mu \frac{x_1 + \mu}{N_1} - \mu \frac{x_1 - \hat{\mu}}{N_2}, \]

\[ x_2'' = x_2 - 2x_1' - \hat{\mu} \frac{x_2}{N_1} - \mu \frac{x_2}{N_2}, \]

where,

\[ N_1 = ((x_1 + \mu)^2 + x_2^2)^{3/2}, \quad N_2 = ((x_1 - \hat{\mu})^2 + x_2^2)^{3/2}. \]

In addition, \( \mu = 0.012277471 \) and \( \hat{\mu} = 1 - \mu \). Initial values are given by

\[ x_1(0) = 0.994, \quad x_1'(0) = 0, \quad x_2(0) = 0, \quad x_2'(0) = -2.001585106. \]

The exact solution is periodic with period \( T = 17.0652166 \). Thus, a measure of the error can be obtained by considering how close we are to the initial position at \( t = T \).

Note that we need to rewrite this system of two second order equations as a system of four first order equations before proceeding to integrate them numerically.

Please keep a record of how many function evaluations are required and how many times steps are accepted and rejected. Try different initial choices of the first step to see if the results will be sensitive to an uninformed choice. Also plot the step size as a function of time.

2. Write a short report on your series of numerical experiments. Looking at the results and the equations, what are the most difficult part of the trajectory to compute?

3. Can you realistically solve this equation using Runge-Kutta and a fixed step size? If so, how many steps would be required?

4. Try to solve the following equations as well with your program:

\[ y_1' = \alpha - y_1 - \frac{4y_1 y_2}{1 + y_1^2}, \]

\[ y_2' = \beta y_1 (1 - \frac{y_2}{1 + y_1^2}), \]

where \( \alpha \) and \( \beta \) are real parameters. Select \( \alpha = 10 \) and explore how the solution will differ for values at, above, and below \( \beta = 3.5 \), which turns out to be a critical value. Select \( y_1(0) = 0, y_2(0) = 2 \), and let \( 0 \leq t \leq 20 \).