Exercises on Catalan and Related Numbers
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19. [1]-[3+] Show that the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \) count the number of elements of the 66 sets \( S_i \), \((a) \leq i \leq (mn)\) given below. We illustrate the elements of each \( S_i \) for \( n = 3 \), hoping that these illustrations will make any undefined terminology clear. (The terms used in (vv)-(yy) are defined in Chapter 7.) Ideally \( S_i \) and \( S_j \) should be proved to have the same cardinality by exhibiting a simple, elegant bijection \( \phi_{ij} : S_i \to S_j \) (so 4290 bijections in all). In some cases the sets \( S_i \) and \( S_j \) will actually coincide, but their descriptions will differ.

(a) triangulations of a convex \((n+2)\)-gon into \( n \) triangles by \( n-1 \) diagonals that do not intersect in their interiors

(b) binary parenthesizations of a string of \( n+1 \) letters

\[
(xx \cdot x) \quad x(xx \cdot x) \quad (x \cdot xx) \quad x(x \cdot xx) \quad xx \cdot xx
\]

(c) binary trees with \( n \) vertices

(d) plane binary trees with \( 2n + 1 \) vertices (or \( n + 1 \) endpoints)

(e) plane trees with \( n + 1 \) vertices

(f) planted (i.e., root has degree one) trivalent plane trees with \( 2n + 2 \) vertices

221
(g) plane trees with $n + 2$ vertices such that the rightmost path of each subtree of the root has even length

(h) lattice paths from $(0, 0)$ to $(n, n)$ with steps $(0, 1)$ or $(1, 0)$, never rising above the line $y = x$

(i) Dyck paths from $(0, 0)$ to $(2n, 0)$, i.e., lattice paths with steps $(1, 1)$ and $(1, -1)$, never falling below the $x$-axis

(j) Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n + 2, 0)$ such that any maximal sequence of consecutive steps $(1, -1)$ ending on the $x$-axis has odd length

(k) Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n + 2, 0)$ with no peaks at height two.
(l) (unordered) pairs of lattice paths with \( n + 1 \) steps each, starting at \((0, 0)\), using steps \((1, 0)\) or \((0, 1)\), ending at the same point, and only intersecting at the beginning and end.

(m) (unordered) pairs of lattice paths with \( n - 1 \) steps each, starting at \((0, 0)\), using steps \((1, 0)\) or \((0, 1)\), ending at the same point, such that one path never arises above the other path.

(n) \( n \) nonintersecting chords joining \( 2n \) points on the circumference of a circle.

(o) ways of connecting \( 2n \) points in the plane lying on a horizontal line by \( n \) nonintersecting arcs, each arc connecting two of the points and lying above the points.

(p) ways of drawing in the plane \( n + 1 \) points lying on a horizontal line \( L \) and \( n \) arcs connecting them such that (\( \alpha \)) the arcs do not pass below \( L \), (\( \beta \)) the graph thus formed is a tree, (\( \gamma \)) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (\( \delta \)) at every vertex, all the arcs exit in the same direction (left or right).

(q) ways of drawing in the plane \( n + 1 \) points lying on a horizontal line \( L \) and \( n \) arcs connecting them such that (\( \alpha \)) the arcs do not pass below \( L \), (\( \beta \)) the graph thus
formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right)

(r) sequences of $n$ 1’s and $n-1$’s such that every partial sum is nonnegative (with $-1$ denoted simply as − below)

![](image)

111111− 11−1− 11−−− 1−1−− 1−1−−−

(s) sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$

111 112 113 122 123

(t) sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq 2i$

12 13 14 23 24

(u) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$

000 001 010 011 012

(v) sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative

0, 0 0, 1 1, −1 1, 0 1, 1

(w) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$

0, 0, 0 0, 1, −1 1, 0, −1 1, −1, 0 2, −1, −1

(x) sequences $a_1, a_2, \ldots, a_n$ of integers such that $0 \leq a_i \leq n - i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$, then $j - i > a_i - a_j$

000 010 100 200 110

(y) sequences $a_1, a_2, \ldots, a_n$ of integers such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$

123 133 223 323 333

(z) sequences $a_1, a_2, \ldots, a_n$ of integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r \leq j - 1$

111 112 113 121 123

224
(aa) equivalence classes \( B \) of words in the alphabet \([n - 1]\) such that any three consecutive letters of any word in \( B \) are distinct, under the equivalence relation \( uijv \sim ujiv \) for any words \( u, v \) and any \( i, j \in [n - 1] \) satisfying \(|i - j| \geq 2\)

\[
\{\emptyset\} \quad \{1\} \quad \{2\} \quad \{12\} \quad \{21\}
\]

(For \( n = 4 \) a representative of each class is given by \( \emptyset, 1, 2, 3, 12, 21, 13, 23, 32, 132, 213, 2132. \)

(bb) partitions \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) with \( \lambda_1 \leq n - 1 \) (so the diagram of \( \lambda \) is contained in an \((n - 1) \times (n - 1)\) square), such that if \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) denotes the conjugate partition to \( \lambda \) then \( \lambda'_i \geq \lambda_i \) whenever \( \lambda_i \geq i \)

\[
(0,0) \quad (1,0) \quad (1,1) \quad (2,1) \quad (2,2)
\]

(cc) permutations \( a_1a_2 \cdots a_{2n} \) of the multiset \( \{1^2, 2^2, \ldots, n^2\} \) such that: (i) the first occurrences of 1, 2, \ldots, \( n \) appear in increasing order, and (ii) there is no subsequence of the form \( \alpha \beta \alpha \beta \)

\[
112233 \quad 112332 \quad 122331 \quad 123321 \quad 122133
\]

(dd) permutations \( a_1a_2 \cdots a_{2n} \) of the set \([2n]\) such that: (i) 1, 3, \ldots, 2n - 1 appear in increasing order, (ii) 2, 4, \ldots, 2n appear in increasing order, and (iii) 2i - 1 appears before 2i, 1 \( \leq i \leq n \)

\[
123456 \quad 123546 \quad 132456 \quad 132546 \quad 135246
\]

(ee) permutations \( a_1a_2 \cdots a_n \) of \([n]\) with longest decreasing subsequence of length at most two (i.e., there does not exist \( i < j < k; a_i > a_j > a_k \)), called \( 321 \)-avoiding permutations

\[
123 \quad 213 \quad 132 \quad 312 \quad 231
\]

(ff) permutations \( a_1a_2 \cdots a_n \) of \([n]\) for which there does not exist \( i < j < k \) and \( a_j < a_k < a_i \) (called \( 312 \)-avoiding permutations)

\[
123 \quad 132 \quad 213 \quad 231 \quad 321
\]

(pp) permutations \( w \) of \([2n]\) with \( n \) cycles of length two, such that the product \((1, 2, \ldots, 2n) \cdot w\) has \( n + 1 \) cycles

\[
(1, 2, 3, 4, 5, 6)(1, 2)(3, 4)(5, 6) = (1)(2, 4, 6)(3)(5)
\]
\[
(1, 2, 3, 4, 5, 6)(1, 2)(3, 6)(4, 5) = (1)(2, 6)(3, 5)(4)
\]
\[
(1, 2, 3, 4, 5, 6)(1, 4)(2, 3)(5, 6) = (1, 3)(2)(4, 6)(5)
\]
\[
(1, 2, 3, 4, 5, 6)(1, 6)(2, 3)(4, 5) = (1, 3, 5)(2)(4, 6)
\]
\[
(1, 2, 3, 4, 5, 6)(1, 6)(2, 5)(3, 4) = (1, 5)(2, 4)(3, 6)
\]

225
(hh) pairs \((u, v)\) of permutations of \([n]\) such that \(u\) and \(v\) have a total of \(n + 1\) cycles, and \(w = (1, 2, \ldots, n)\)

\[
\begin{align*}
(1)(2)(3) \cdot (1, 2, 3) & \quad (1, 2, 3) \cdot (1)(2)(3) \\
(1, 2)(3) \cdot (1, 2, 3) & \quad (1)(2, 3) \cdot (1, 2)(3)
\end{align*}
\]

(ii) permutations \(a_1 a_2 \cdots a_n\) of \([n]\) that can be put in increasing order on a single stack, defined recursively as follows: If \(\emptyset\) is the empty sequence, then let \(S(\emptyset) = \emptyset\). If \(w = uv\) is a sequence of distinct integers with largest term \(n\), then \(S(w) = S(u)S(v)\). A stack-sortable permutation \(w\) is one for which \(S(w) = w\).

\[
\begin{array}{ccccc}
4 & 1 & 2 & 3 & 123 \\
12 & 3 \\
2 & 4 & 1234 \\
123 & 132 & 213 & 312 & 321
\end{array}
\]

For example,

(jj) permutations \(a_1 a_2 \cdots a_n\) of \([n]\) that can be put in increasing order on two parallel queues. Now the picture is

\[
\begin{array}{cccccccc}
123 & 132 & 213 & 231 & 312
\end{array}
\]

(kk) fixed-point free involutions \(w\) of \([2n]\) such that if \(i < j < k < l\) and \(w(i) = k\), then \(w(j) \neq l\) (in other words, 3412-avoiding fixed-point free involutions)

\[
\begin{align*}
(12)(34)(56) & \quad (14)(23)(56) \\
(12)(36)(45) & \quad (16)(23)(45) \\
(16)(25)(34)
\end{align*}
\]

(ll) cycles of length \(2n + 1\) in \(S_{2n+1}\) with descent set \(\{n\}\)

\[
\begin{align*}
2371456 & \quad 2571346 \\
3471256 & \quad 3671245 \\
5671234
\end{align*}
\]

(mm) Baxter permutations (as defined in Exercise 55) of \([2n]\) or of \([2n + 1]\) that are reverse alternating (as defined at the end of Section 3.16) and whose inverses are reverse alternating

\[
\begin{align*}
132546 & \quad 153426 & \quad 354612 & \quad 561324 & \quad 563412 \\
1325476 & \quad 1327564 & \quad 1534276 & \quad 1735462 & \quad 1756342
\end{align*}
\]

226
(nn) permutations \(w\) of \([n]\) such that if \(w\) has \(\ell\) inversions then for all pairs of sequences \((a_1, a_2, \ldots, a_\ell), (b_1, b_2, \ldots, b_\ell) \in [n - 1]^\ell\) satisfying
\[
w = s_{a_1}s_{a_2} \cdots s_{a_\ell} = s_{b_1}s_{b_2} \cdots s_{b_\ell},
\]
where \(s_j\) is the adjacent transposition \((j, j + 1)\), we have that the \(\ell\)-element multisets \(\{a_1, a_2, \ldots, a_\ell\}\) and \(\{b_1, b_2, \ldots, b_\ell\}\) are equal (thus, for example, \(w = 321\) is not counted, since \(w = s_1s_2s_1 = s_2s_1s_2\), and the multisets \(\{1, 2, 1\}\) and \(\{2, 1, 2\}\) are not equal)
\[
123 \quad 132 \quad 213 \quad 231 \quad 312
\]

(oo) permutations \(w\) of \([n]\) with the following property: Suppose that \(w\) has \(\ell\) inversions, and let
\[
R(w) = \{(a_1, \ldots, a_\ell) \in [n - 1]^\ell : w = a_1a_2 \cdots a_\ell\},
\]
where \(s_j\) is as in (nn). Then
\[
\sum_{(a_1, \ldots, a_\ell) \in R(w)} a_1a_2 \cdots a_\ell = \ell!.
\]
\[
R(123) = \{\emptyset\}, \quad R(213) = \{(1)\}, \quad R(231) = \{(1, 2)\}\]
\[
R(312) = \{(2, 1)\}, \quad R(321) = \{(1, 2, 1), (2, 1, 2)\}\]

(pp) noncrossing partitions of \([n]\), i.e., partitions \(\pi = \{B_1, \ldots, B_k\} \in \Pi_n\) such that if \(a < b < c < d\) and \(a, c \in B_i\) and \(b, d \in B_j\), then \(i = j\)
\[
123 \quad 12-3 \quad 13-2 \quad 23-1 \quad 1-2-3
\]

(qq) partitions \(\{B_1, \ldots, B_k\}\) of \([n]\) such that if the numbers \(1, 2, \ldots, n\) are arranged in order around a circle, then the convex hulls of the blocks \(B_1, \ldots, B_k\) are pairwise disjoint

\[
\begin{array}{c}
. \\
. \\
. \\
. \\
. \\
. \\
. \\
\end{array}
\]

(rr) noncrossing Murasakii diagrams with \(n\) vertical lines

\[
\begin{array}{c}
| \\
| \\
| \\
| \\
| \\
\end{array}
\]

(ss) noncrossing partitions of some set \([k]\) with \(n+1\) blocks, such that any two elements of the same block differ by at least three
\[
1-2-3-4 \quad 14-2-3-5 \quad 15-2-3-4 \quad 25-1-3-4 \quad 16-25-3-4
\]

(tt) noncrossing partitions of \([2n + 1]\) into \(n + 1\) blocks, such that no block contains two consecutive integers
\[
137-46-2-5 \quad 1357-2-4-6 \quad 157-24-3-6 \quad 17-246-3-5 \quad 17-26-35-4
\]

227
(uu) **nonnesting partitions** of \([n]\), i.e., partitions of \([n]\) such that if \(a, e\) appear in a block \(B\) and \(b, d\) appear in a different block \(B'\) where \(a < b < d < e\), then there is a \(c \in B\) satisfying \(b < c < d\)

\[
\begin{array}{cccccc}
123 & 12-3 & 13-2 & 23-1 & 1-2-3 \\
\end{array}
\]

(The unique partition of \([4]\) that isn’t nonnesting is \(14-23\).)

(vv) Young diagrams that fit in the shape \((n-1, n-2, \ldots, 1)\)

\[
\emptyset \quad \square \quad \square \quad \square \quad \square \quad \square
\]

(ww) standard Young tableaux of shape \((n, n)\) (or equivalently, of shape \((n, n-1)\))

\[
\begin{array}{cccccc}
123 & 124 & 125 & 134 & 135 \\
456 & 356 & 346 & 256 & 246 \\
\end{array}
\]

or

\[
\begin{array}{cccccc}
123 & 124 & 125 & 134 & 135 \\
45 & 35 & 34 & 25 & 24 \\
\end{array}
\]

(xx) pairs \((P, Q)\) of standard Young tableaux of the same shape, each with \(n\) squares and at most two rows

\[
(123, 123) \quad \left(\begin{array}{cc}
12 & 12 \\
3 & 3 \\
\end{array}\right) \quad \left(\begin{array}{cc}
12 & 13 \\
3 & 2 \\
\end{array}\right) \quad \left(\begin{array}{cc}
13 & 12 \\
2 & 3 \\
\end{array}\right) \quad \left(\begin{array}{cc}
13 & 13 \\
2 & 2 \\
\end{array}\right)
\]

(yy) column-strict plane partitions of shape \((n-1, n-2, \ldots, 1)\), such that each entry in the \(i\)th row is equal to \(n - i\) or \(n - i + 1\)

\[
\begin{array}{cccccc}
33 & 33 & 32 & 32 & 22 \\
2 & 1 & 2 & 1 & 1 \\
\end{array}
\]

(zz) convex subsets \(S\) of the poset \(\mathbb{Z} \times \mathbb{Z}\), up to translation by a diagonal vector \((m, m)\), such that if \((i, j) \in S\) then \(0 < i - j < n\).

\[
\emptyset \quad \{(1, 0)\} \quad \{(2, 0)\} \quad \{(1, 0), (2, 0)\} \quad \{(2, 0), (2, 1)\}
\]

(aaa) linear extensions of the poset \(2 \times \mathbf{n}\)

\[
\[
\begin{array}{cccccc}
123456 \\
123546 \\
132456 \\
132546 \\
135246 \\
\end{array}
\]

228
(bbb) order ideals of $\text{Int}(n - 1)$, the poset of intervals of the chain $n - 1$

(bbb) order ideals of $\text{Int}(n - 1)$, the poset of intervals of the chain $n - 1$

(ccc) order ideals of the poset $A_n$ obtained from the poset $(n - 1) \times (n - 1)$ by adding the relations $(i, j) < (j, i)$ if $i > j$ (see Figure 5 for the Hasse diagram of $A_4$)

$$
\emptyset, \ a, \ b, \ ab, \ abc
$$

(ddd) nonisomorphic $n$-element posets with no induced subposet isomorphic to $2 + 2$ or $3 + 1$

(eee) nonisomorphic $(n + 1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element

(fff) relations $R$ on $[n]$ that are reflexive ($iRi$), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \leq i < j < k \leq n$ and $iRk$, then $iRj$ and $jRk$ (in the example below we write
$ij$ for the pair $(i, j)$, and we omit the pairs $ii$)

$$\emptyset \ \{12, 21\} \ \{23, 32\} \ \{12, 21, 23, 32\} \ \{12, 21, 13, 31, 23, 32\}$$

(ggg) joining some of the vertices of a convex $(n - 1)$-gon by disjoint line segments, and circling a subset of the remaining vertices

$$\begin{array}{c}
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\end{array}$$

(hhh) ways to stack coins in the plane, the bottom row consisting of $n$ consecutive coins

(iii) $n$-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\ldots a_n1$, each $a_i$ divides the sum of its two neighbors

$$14321 \ \ 13521 \ \ 13231 \ \ 12531 \ \ 12341$$

(iii) $n$-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0

$$000 \ \ \ 013 \ \ \ 022 \ \ \ 112 \ \ \ 233$$

(kkk) $n$-element subsets $S$ of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \geq j$ and there is a lattice path from $(0, 0)$ to $(i, j)$ with steps $(0, 1), (1, 0)$, and $(1, 1)$ that lies entirely inside $S$

$$\{(0, 0), (1, 0), (2, 0)\} \ \ \{(0, 0), (1, 0), (1, 1)\} \ \ \{(0, 0), (1, 0), (2, 1)\}$$

$$\{(0, 0), (1, 1), (2, 1)\} \ \ \{(0, 0), (1, 1), (2, 2)\}$$

(lili) regions into which the cone $x_1 \geq x_2 \geq \cdots \geq x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leq i < j \leq n$ (the diagram below shows the situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$)

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```

230
\[
\begin{array}{cccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 2 & 1 & 5 & 1 & 2 & 3 & 1 & 3 & 2 & 1 & 5 & 4 \\
2 & 5 & 1 & 4 & 4 & 1 & 5 & 2 & 2 & 5 & 1 & 4 & 4 & 1 \\
3 & 2 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
1 & 5 & 2 & 2 & 5 & 1 & 4 & 4 & 1 & 5 & 1 & 2 & 3 & 1 \\
2 & 3 & 1 & 3 & 2 & 1 & 5 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Figure 6: The frieze pattern corresponding to the sequence \(1, 3, 2, 1, 5, 1, 2, 3\)

(mmm) positive integer sequences \(a_1, a_2, \ldots, a_{n+2}\) for which there exists an integer array
(necessarily with \(n + 1\) rows)

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & a_1 & a_2 & a_3 & \cdots & a_{n+2} & a_1 & a_2 & \cdots & a_{n-1} \\
b_1 & b_2 & b_3 & \cdots & b_{n+2} & b_1 & b_2 & \cdots & b_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_1 & r_2 & r_3 & \cdots & r_{n+2} & r_1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\end{array}
\]

(54)

such that any four neighboring entries in the configuration \(r \atop s, t \atop u\) satisfy \(st = ru + 1\)

(an example of such an array for \((a_1, \ldots, a_8) = (1, 3, 2, 1, 5, 1, 2, 3)\) (necessarily
unique) is given by Figure 6):

\[
\begin{array}{cccccccc}
12213 & 22131 & 21312 & 13122 & 31221 \\
\end{array}
\]

(nnn) \(n\)-tuples \((a_1, \ldots, a_n)\) of positive integers such that the tridiagonal matrix

\[
\begin{bmatrix}
    a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
    1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\
    0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\
    0 & 0 & 0 & 0 & \cdots & 1 & a_n
\end{bmatrix}
\]

is positive definite with determinant one

\[
\begin{array}{ccccccc}
131 & 122 & 221 & 213 & 312 \\
\end{array}
\]

20. (a) [2+] Let \(m, n\) be integers satisfying \(1 \leq n < m\). Show by a simple bijection that
the number of lattice paths from \((1, 0)\) to \((m, n)\) with steps \((0, 1)\) and \((1, 0)\) that
intersect the line \(y = x\) in at least one point is equal to the number of lattice
paths from \((0, 1)\) to \((m, n)\) with steps \((0, 1)\) and \((1, 0)\).
(b) [2-] Deduce that the number of lattice paths from $(0, 0)$ to $(m, n)$ with steps $(1, 0)$ and $(0, 1)$ that intersect the line $y = x$ only at $(0, 0)$ is given by $\frac{m-n}{m+n} \binom{m+n}{m}$.

(c) [1+] Show from (b) that the number of lattice paths from $(0, 0)$ to $(n, n)$ with steps $(1, 0)$ and $(0, 1)$ that never rise above the line $y = x$ is given by the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. (This gives a direct combinatorial proof of interpretation (h) of $C_n$ in Exercise 19.)

21. (a) [2+] Let $X_n$ be the set of all $\binom{2n}{n}$ lattice paths from $(0, 0)$ to $(n, n)$ with steps $(0, 1)$ and $(1, 0)$. Define the \textit{excedance} (also spelled “exceedance”) of a path $P \in X_n$ to be the number of $i$ such that at least one point $(i, i')$ of $P$ lies above the line $y = x$ (i.e., $i' > i$). Show that the number of paths in $X_n$ with excedance $j$ is independent of $j$.

(b) [1] Deduce that the number of $P \in X_n$ that never rise above the line $y = x$ is given by the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ (a direct proof of interpretation (h) of $C_n$ in Exercise 19). Compare with Example 5.3.11, which also gives a direct combinatorial interpretation of $C_n$ when written in the form $\frac{1}{n+1} \binom{2n+1}{n}$ (as well as in the form $\frac{1}{2n+1} \binom{2n+1}{n}$).

22. [2+] Show (bijectively if possible) that the number of lattice paths from $(0, 0)$ to $(2n, 2n)$ with steps $(1, 0)$ and $(0, 1)$ that avoid the points $(2i - 1, 2i - 1)$, $1 \leq i \leq n$, is equal to the Catalan number $C_{2n}$.

23. [3-] Consider the following chess position.

Black is to make 19 consecutive moves, after which White checkmates Black in one move. Black may not move into check, and may not check White (except possibly on his last move). Black and White are \textit{cooperating} to achieve the aim of checkmate. (In chess problem parlance, this problem is called a \textit{serieshelpmate in 19}.) How many different solutions are there?
24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, ...

25. [2]–[5] Show that the Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \) has the algebraic interpretations given below.

(a) number of two-sided ideals of the algebra of all \((n - 1) \times (n - 1)\) upper triangular matrices over a field

(b) dimension of the space of invariants of \( \text{SL}(2, \mathbb{C}) \) acting on the \( 2n \)th tensor power \( T^{2n}(V) \) of its “defining” two-dimensional representation \( V \)

(c) dimension of the irreducible representation of the symplectic group \( \text{Sp}(2(n-1), \mathbb{C}) \) (or Lie algebra \( \mathfrak{sp}(2(n-1), \mathbb{C}) \)) with highest weight \( \lambda_{n-1} \), the \((n-1)\)st fundamental weight

(d) dimension of the primitive intersection homology (say with real coefficients) of the toric variety associated with a (rationally embedded) \( n \)-dimensional cube

(e) the generic number of \( \text{PGL}(2, \mathbb{C}) \) equivalence classes of degree \( n \) rational maps with a fixed branch set

(f) number of translation conjugacy classes of degree \( n + 1 \) monic polynomials in one complex variable, all of whose critical points are fixed

(g) dimension of the algebra (over a field \( K \)) with generators \( \epsilon_1, \ldots, \epsilon_{n-1} \) and relations

\[
\begin{align*}
\epsilon_i^2 &= \epsilon_i \\
\beta \epsilon_i \epsilon_j \epsilon_i &= \epsilon_i, \text{ if } |i - j| = 1 \\
\epsilon_i \epsilon_j &= \epsilon_j \epsilon_i, \text{ if } |i - j| \geq 2,
\end{align*}
\]

where \( \beta \) is a nonzero element of \( K \)

(h) number of \( \oplus \)-sign types indexed by \( A^+_n \) (the set of positive roots of the root system \( A_{n-1} \))

(i) Let the symmetric group \( \mathfrak{S}_n \) act on the polynomial ring \( A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) by \( w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)}) \) for all \( w \in \mathfrak{S}_n \).

Let \( I \) be the ideal generated by all invariants of positive degree, i.e.,

\[
I = \langle f \in A : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0 \rangle.
\]

Then (conjecturally) \( C_n \) is the dimension of the subspace of \( A/I \) affording the sign representation, i.e.,

\[
C_n = \dim \{ f \in A/I : w \cdot f = (\text{sgn } w) f \text{ for all } f \in \mathfrak{S}_n \}.
\]

26. (a) [3] Let \( D \) be a Young diagram of a partition \( \lambda \), as defined in Section 1.3. Given a square \( s \) of \( D \) let \( t \) be the lowest square in the same column as \( s \), and let \( u \) be the rightmost square in the same row as \( s \). Let \( f(s) \) be the number of paths from
t to u that stay within \( D \), and such that each step is one square to the north or one square to the east. Insert the number \( f(s) \) in square \( s \), obtaining an array \( A \). For instance, if \( \lambda = (5, 4, 3, 3) \) then \( A \) is given by

\[
\begin{bmatrix}
16 & 7 & 2 & 1 & 1 \\
6 & 3 & 1 & 1 \\
3 & 2 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

Let \( M \) be the largest square subarray (using consecutive rows and columns) of \( A \) containing the upper left-hand corner. Regard \( M \) as a matrix. For the above example we have

\[
M = \begin{bmatrix} 16 & 7 & 2 \\ 6 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}.
\]

Show that \( \det M = 1 \).

(b) [2] Find the unique sequence \( a_0, a_1, \ldots \) of real numbers such that for all \( n \geq 0 \) we have

\[
\begin{align*}
\det \begin{bmatrix}
a_0 & a_1 & \cdots & a_n \\
a_1 & a_2 & \cdots & a_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
a_n & a_{n+1} & \cdots & a_{2n} \\
\end{bmatrix} &= \det \begin{bmatrix}
a_1 & a_2 & \cdots & a_n \\
a_2 & a_3 & \cdots & a_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
a_n & a_{n+1} & \cdots & a_{2n-1} \\
\end{bmatrix} = 1.
\end{align*}
\]

(When \( n = 0 \) the second matrix is empty and by convention has determinant one.)

27. (a) [3–] Let \( V_n \) be a real vector space with basis \( x_0, x_1, \ldots, x_n \) and scalar product defined by \( \langle x_i, x_j \rangle = C_{i+j} \), the \((i + j)\)-th Catalan number. It follows from Exercise 26(b) that this scalar product is positive definite, and therefore \( V \) has an orthonormal basis. Is there an orthonormal basis for \( V_n \) whose elements are integral linear combinations of the \( x_i \)'s ?

(b) [3–] Same as (a), except now \( \langle x_i, x_j \rangle = C_{i+j+1} \).

(c) [5–] Investigate the same question for the matrices \( M \) of Exercise 26(a) (so \( \langle x_i, x_j \rangle = M_{ij} \)) when \( \lambda \) is self-conjugate (so \( M \) is symmetric).

28. (a) [3–] Suppose that real numbers \( x_1, x_2, \ldots, x_d \) are chosen uniformly and independently from the interval \([0, 1]\). Show that the probability that the sequence \( x_1, x_2, \ldots, x_d \) is convex (i.e., \( x_i \leq \frac{1}{2}(x_{i-1} + x_{i+1}) \) for \( 2 \leq i \leq d-1 \)) is \( C_{d-1}/(d-1)!^2 \), where \( C_{d-1} \) denotes a Catalan number.
(b) [3–] Let $\mathcal{C}_d$ denote the set of all points $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ such that $0 \leq x_i \leq 1$ and the sequence $x_1, x_2, \ldots, x_d$ is convex. It is easy to see that $\mathcal{C}_d$ is a $d$-dimensional convex polytope, called the convexotope. Show that the vertices of $\mathcal{C}_d$ consist of the points

$$\left(1, \frac{i-1}{j}, \frac{i-2}{j}, \ldots, \frac{1}{j}, 0, 0, \ldots, 0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\right)$$

(with at least one 0 coordinate), together with $(1, 1, \ldots, 1)$ (so $(d+1)/2 + 1$ vertices in all). For instance, the vertices of $\mathcal{C}_3$ are $(0, 0, 0)$, $(0, 0, 1)$, $(0, \frac{1}{2}, 1)$, $(1, 0, 0)$, $(1, \frac{1}{2}, 0)$, $(1, 0, 1)$, $(1, 1, 1)$.

(c) [3] Show that the Ehrhart quasi-polynomial $\ell(\mathcal{C}_d, n)$ of $\mathcal{C}_d$ (as defined in Section 4.6) is given by

$$y_d := \sum_{n \geq 0} \ell(\mathcal{C}_d, n) x^n$$

$$= \frac{1}{1-x} \left( \sum_{r=1}^{d} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-r]!} - \sum_{r=1}^{d-1} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-1-r]!} \right),$$

where $[i] = 1 - x^i$, $[i!] = [1][2] \cdots [i]$, and $*$ denotes Hadamard product. For instance,

$$y_1 = \frac{1}{(1-x)^2}$$

$$y_2 = \frac{1+x}{(1-x)^3}$$

$$y_3 = \frac{1+2x+3x^2}{(1-x)^3(1-x^2)}$$

$$y_4 = \frac{1+3x+9x^2+12x^3+11x^4+3x^5+x^6}{(1-x)^2(1-x^2)^2(1-x^3)}$$

$$y_5 = \frac{1+4x+14x^2+34x^3+63x^4+80x^5+87x^6+68x^7+42x^8+20x^9+7x^{10}}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}.$$

Is there a simpler formula than (56) for $\ell(\mathcal{C}_d, n)$ or $y_d$?

29. [3] Suppose that $n+1$ points are chosen uniformly and independently from inside a square. Show that the probability that the points are in convex position (i.e., each point is a vertex of the convex hull of all the points) is $(C_n/n!)^2$.

30. [3–] Let $f_n$ be the number of partial orderings of the set $[n]$ that contain no induced subposets isomorphic to $3+1$ or $2+2$. (This exercise is the labelled analogue of Exercise 19(ddd). A mention of the solution to this exercise, such posets are called semiorders.) Let $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ be the generating function for Catalan numbers. Show that

$$\sum_{n \geq 0} f_n \frac{x^n}{n!} = C(1 - e^{-x}),$$

the composition of $C(x)$ with the series $1 - e^{-x} = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \cdots$. 

235
31. (a) [3–] Let $\mathcal{P}$ denote the convex hull in $\mathbb{R}^{d+1}$ of the origin together with all vectors $e_i - e_j$, where $e_i$ is the $i$th unit coordinate vector and $i < j$. Thus $\mathcal{P}$ is a $d$-dimensional convex polytope. Show that the relative volume of $\mathcal{P}$ (as defined in Section 4.6) is equal to $C_d/d!$, where $C_d$ denotes a Catalan number.

(b) [3] Let $i(\mathcal{P}, n)$ denote the Ehrhart polynomial of $\mathcal{P}$. Find a combinatorial interpretation of the coefficients of the $i$-Eulerian polynomial (in the terminology of Section 4.3)

$$(1 - x)^{d+1} \sum_{n \geq 0} i(\mathcal{P}, n) x^n.$$ 

32. (a) [3–] Define a partial order $T_n$ on the set of all binary bracketings (parenthesizations) of a string of length $n + 1$ as follows. We say that $v$ covers $u$ if $u$ contains a subexpression $(xy)z$ (where $x$, $y$, $z$ are bracketed strings) and $v$ is obtained from $u$ by replacing $(xy)z$ with $x(yz)$. For instance, $((a^2 \cdot a)(a^2 \cdot a^2))$ is covered by $((a \cdot a^2)(a^2 \cdot a^2))(a^2 \cdot a^2)$, $(a^2(a \cdot a^2))(a^2 \cdot a^2))$, and $(a^2 \cdot a)(a^2(a^2 \cdot a^2)))$.

Figures 7 and 8 show the Hasse diagrams of $T_3$ and $T_4$. (In Figure 8, we have encoded the binary bracketing by a string of four $+$'s and four $-$'s, where $+$ stands for a left parenthesis and $-$ for the letter $a$, with the last $a$ omitted.) Let $U_n$ be the poset of all integer vectors $(a_1, a_2, \ldots, a_n)$ such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$ then $a_j \leq a_i$, ordered coordinatewise. Show that $T_n$ and $U_n$ are isomorphic posets.

(b) [2] Deduce from (a) that $T_n$ is a lattice (called the Tamari lattice).

33. Let $C$ be a convex $n$-gon. Let $\mathcal{S}$ be the set of all sets of diagonals of $C$ that do not intersect in the interior of $C$. Partially order the element of $\mathcal{S}$ by inclusion, and add a $\hat{1}$. Call the resulting poset $A_n$.

(a) [3–] Show that $A_n$ is a simplicial Eulerian lattice of rank $n - 2$, as defined in Section 3.14.

(b) [3] Show in fact that $A_n$ is the lattice of faces of an $(n - 3)$-dimensional convex polytope $Q_n$.

(c) [3–] Find the number $W_i = W_i(n)$ of elements of $A_n$ of rank $i$. Equivalently, $W_i$ is the number of ways to draw $i$ diagonals of $C$ that do not intersect in their interiors. Note that by Proposition 2.1, $W_i(n)$ is also the number of plane trees with $n + i$ vertices and $n - 1$ endpoints such that no vertex has exactly one successor.
Figure 8: The Tamari lattice $T_4$
(d) [3+] Define
\[ \sum_{i=0}^{n-3} W_i(x - 1)^{n-i-3} = \sum_{i=0}^{n-3} h_i x^{n-3-i}, \quad (58) \]
as in equation (3.44). The vector \((h_0, \ldots, h_{n-3})\) is called the \(h\)-vector of \(A_n\) (or of the polytope \(Q_n\)). Find an explicit formula for each \(h_i\).

34. There are many possible \(q\)-analogues of Catalan numbers. In (a) we give what is perhaps the most natural “combinatorial” \(q\)-analogue, while in (b) we give the most natural “explicit formula” \(q\)-analogue. In (c) we give an interesting extension of (b), while (d) and (e) are concerned with another special case of (c).

(a) [2+] Let
\[
C_n(q) = \sum_{P} q^{A(P)},
\]
where the sum is over all lattice paths \(P\) from \((0,0)\) to \((n,n)\) with steps \((1,0)\) and \((0,1)\), such that \(P\) never rises above the line \(y = x\), and where \(A(P)\) is the area under the path (and above the \(x\)-axis). Note that by Exercise 19(h), we have \(C_n(1) = C_n\). (It is interesting to see what statistic corresponds to \(A(P)\) for many of the other combinatorial interpretations of \(C_n\) given in Exercise 19.) For instance, \(C_0(q) = C_1(q) = 1\), \(C_2(q) = 1 + q\), \(C_3(q) = 1 + q + 2q^2 + q^3\), \(C_4(q) = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6\). Show that
\[
C_{n+1}(q) = \sum_{i=0}^{n} C_i(q) C_{n-i}(q) q^{(i+1)(n-i)}.
\]
Deduce that if \(\tilde{C}_n(q) = q^{n(n)} C_n(1/q)\), then the generating function
\[
F(x) = \sum_{n\geq0} \tilde{C}_n(q)x^n
\]
satisfies
\[
x F(x) F(qx) - F(x) + 1 = 0.
\]
From this we get the continued fraction expansion
\[
F(x) = \frac{1}{x - \frac{q x}{1 - \frac{q^2 x}{1 - \cdots}}}. \quad (59)
\]

(b) [2+] Define
\[
c_n(q) = \frac{1}{(n+1)} \binom{2n}{n}.
\]
For instance, \( c_0(q) = c_1(q) = 1 \), \( c_2(q) = 1 + q^2 \), \( c_3(q) = 1 + q^2 + q^3 + q^4 + q^6 \), \( c_4(q) = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12} \). Show that

\[
c_n(q) = \sum_w q^{\text{maj}(w)},
\]

where \( w \) ranges over all sequences \( a_1a_2\cdots a_{2n} \) of \( n \) 1’s and \( n-1 \)’s such that each partial sum is nonnegative, and where

\[
\text{maj}(w) = \sum_{\{i : a_i > a_{i+1}\}} i,
\]

the major index of \( w \).

(c) \( \text{[3–] Let } t \text{ be a parameter, and define}

\[
c_n(t; q) = \frac{1}{(n + 1)} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i+1} q^{2i+it}.
\]

Show that

\[
c_n(t; q) = \sum_w q^{\text{maj}(w) + (t-1)\text{des}(w)},
\]

where \( w \) ranges over the same set as in (b), and where

\[
\text{des}(w) = \#\{i : a_i > a_{i+1}\},
\]

the number of descents of \( w \). (Hence \( c_n(1; q) = c_n(q) \).)

(d) \( \text{[3–] Show that}

\[
c_n(0; q) = \frac{1 + q}{1 + q^n} c_n(q).
\]

For instance, \( c_0(0; q) = c_1(0; q) = 1 \), \( c_2(0; q) = 1 + q \), \( c_3(0; q) = 1 + q + q^2 + q^3 + q^4 \), \( c_4(0; q) = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + q^7 + q^8 + q^9 \).

(e) \( \text{[3+] Show that the coefficients of } c_n(0; q) \text{ are unimodal, i.e., if } c_n(0; q) = \sum b_i q^i, \text{ then for some } j \text{ we have } b_0 \leq b_1 \leq \cdots \leq b_j \geq b_{j+1} \geq b_{j+2} \geq \cdots. \text{ (In fact, we can take } j = \lfloor \frac{1}{2} \deg c_n(0; q) \rfloor = \lfloor \frac{1}{2} (n-1)^2 \rfloor \).}

35. Let \( Q_n \) be the poset of direct-sum decompositions of an \( n \)-dimensional vector space \( V_n \) over the field \( \mathbb{F}_q \), as defined in Example 5.5.2(b). Let \( \overline{Q}_n \) denote \( Q_n \) with a \( \overline{0} \) adjoined, and let \( \mu_n(q) = \mu_{\overline{Q}_n}(0, \overline{1}) \). Hence by (5.74) we have

\[
- \sum_{n \geq 1} \mu_n(q) \frac{x^n}{q(n)!} = \log \sum_{n \geq 0} \frac{x^n}{q(n)!}.
\]

(a) \( \text{[3–] Show that}

\[
\mu_n(q) = \frac{1}{n} (-1)^n (q - 1)(q^2 - 1) \cdots (q^{n-1} - 1) P_n(q),
\]

239
where $P_n(q)$ is a polynomial in $q$ of degree $\binom{n}{2}$ with nonnegative integral coefficients, satisfying $P_n(1) = \binom{2n-1}{n}$. For instance,
\begin{align*}
P_1(q) &= 1 \\
P_2(q) &= 2 + q \\
P_3(q) &= 3 + 3q + 3q^2 + q^3 \\
P_4(q) &= (2 + 2q^2 + q^3)(2 + 2q + q^2 + q^3).
\end{align*}

(b) Show that
\[ \exp \sum_{n \geq 1} q^n P_n(1/q) \frac{x^n}{n} = \sum_{n \geq 1} q^n C_n(1/q) x^n, \]
where $C_n(q)$ is the $q$-Catalan polynomial defined in Exercise 34(a).

36. (a) $[2+]$ The Narayana numbers $N(n, k)$ are defined by
\[ N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}. \]

Let $X_{nk}$ be the set of all sequences $w = w_1 w_2 \cdots w_{2n}$ of $n$ 1’s and $n - 1$’s with all partial sums nonnegative, such that
\[ k = \#\{j : w_j = 1, w_{j+1} = -1\}. \]

Give a combinatorial proof that $N(n, k) = \#X_{nk}$. Hence by Exercise 19(r), there follows
\[ \sum_{k=1}^{n} N(n, k) = C_n. \]

(It is interesting to find for each of the combinatorial interpretations of $C_n$ given by Exercise 19 a corresponding decomposition into subsets counted by Narayana numbers.)

(b) $[2+]$ Let $F(x, t) = \sum_{n \geq 1} \sum_{k \geq 1} N(n, k) x^n t^k$. Using the combinatorial interpretation of $N(n, k)$ given in (a), show that
\[ x F^2 + (xt + x - 1) F + xt = 0, \]
so
\[ F(x, t) = \frac{1 - x - xt - \sqrt{(1 - x - xt)^2 - 4x^2t}}{2x}. \]

37. $[2+]$ The Motzkin numbers $M_n$ are defined by
\[ \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}. \]

\[ = 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + 323x^8 + 835x^9 + 2188x^{10} + \ldots. \]

Show that $M_n = \Delta^n C_1$ and $C_n = \Delta^{2n} M_0$, where $C_n$ denotes a Catalan number.
38. [3–] Show that the Motzkin number $M_n$ has the following combinatorial interpretations. (See Exercise 46(b) for an additional interpretation.)

(a) Number of ways of drawing any number of nonintersecting chords among $n$ points on a circle.

(b) Number of walks on $\mathbb{N}$ with $n$ steps, with steps $-1$, $0$, or $1$, starting and ending at $0$.

(c) Number of lattice paths from $(0,0)$ to $(n,n)$, with steps $(0,2)$, $(2,0)$, and $(1,1)$, never rising above the line $y = x$.

(d) Number of paths from $(0,0)$ to $(n,n)$ with steps $(1,0)$, $(1,1)$, and $(1,-1)$, never going below the $x$-axis. Such paths are called Motzkin paths.

(e) Number of pairs $1 \leq a_1 < \cdots < a_k \leq n$ and $1 \leq b_1 < \cdots < b_k \leq n$ of integer sequences such that $a_i \leq b_i$ and every integer in the set $[n]$ appears at least once among the $a_i$'s and $b_i$'s.

(f) Number of ballot sequences (as defined in Corollary 2.3(ii)) $(a_1, \ldots, a_{2n+2})$ such that we never have $(a_{i-1}, a_i, a_{i+1}) = (1, -1, 1)$.

(g) Number of plane trees with $n/2$ edges, allowing “half edges” that have no successors and count as half an edge.

(h) Number of plane trees with $n + 1$ edges in which no vertex, the root excepted, has exactly one successor.

(i) Number of plane trees with $n$ edges in which every vertex has at most two successors.

(j) Number of binary trees with $n - 1$ edges such that no two consecutive edges slant to the right.

(k) Number of plane trees with $n + 1$ vertices such that every vertex of odd height (with the root having height 0) has at most one successor.

(l) Number of noncrossing partitions $\pi = \{B_1, \ldots, B_k\}$ of $[n]$ (as defined in Exercise 3.68) such that if $B_i = \{b\}$ and $a < b < c$, then $a$ and $c$ appear in different blocks of $\pi$.

(m) Number of noncrossing partitions $\pi$ of $[n+1]$ such that no block of $\pi$ contains two consecutive integers.

39. [3–] The Schröder numbers $r_n$ and $s_n$ were defined in Section 2. Show that they have the following combinatorial interpretations.

(a) $s_{n-1}$ is the total number of bracketings (parenthesizations) of a string of $n$ letters.

(b) $s_{n-1}$ is the number of plane trees with no vertex of degree one and with $n$ endpoints.

(c) $r_{n-1}$ is the number of plane trees with $n$ vertices and with a subset of the endpoints circled.
(d) $s_n$ is the number of binary trees with $n$ vertices and with each right edge colored either red or blue.

(e) $s_n$ is the number of lattice paths in the $(x, y)$ plane from $(0, 0)$ to the $x$-axis using steps $(1, k)$, where $k \in \mathbb{P}$ or $k = -1$, never passing below the $x$-axis, and with $n$ steps of the form $(1, -1)$.

(f) $s_n$ is the number of lattice paths in the $(x, y)$ plane from $(0, 0)$ to $(n, n)$ using steps $(k, 0)$ or $(0, 1)$ with $k \in \mathbb{P}$, and never passing above the line $y = x$.

(g) $r_{n-1}$ is the number of parallelogram polyominos (defined in the solution to Exercise 19(I)) of perimeter $2n$ with each column colored either black or white.

(h) $s_n$ is the number of ways to draw any number of diagonals of a convex $(n+2)$-gon that do not intersect in their interiors.

(i) $s_n$ is the number of sequences $i_1i_2 \cdots i_k$, where $i_j \in \mathbb{P}$ or $i_j = -1$ (and $k$ can be arbitrary), such that $n = \# \{ j : i_j = -1 \}$, $i_1 + i_2 + \cdots + i_j \geq 0$ for all $j$, and $i_1 + i_2 + \cdots + i_k = 0$.

(j) $r_n$ is the number of lattice paths from $(0, 0)$ to $(n, n)$, with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$, that never rise above the line $y = x$.

(k) $r_{n-1}$ is the number of $n \times n$ permutation matrices $P$ with the following property: We can eventually reach the all 1’s matrix by starting with $P$ and continually replacing a 0 by a 1 if that 0 has at least two adjacent 1’s, where an entry $a_{ij}$ is defined to be adjacent to $a_{i\pm 1,j}$ and $a_{i,j\pm 1}$.

(l) Let $u = u_1 \cdots u_k \in \mathcal{S}_k$. We say that a permutation $w = w_1 \cdots w_n \in \mathcal{S}_n$ is $u$-avoiding if no subsequence $w_{a_1}, \ldots, w_{a_k}$ (with $a_1 < \cdots < a_k$) is in the same relative order as $u$, i.e., $u_i < u_j$ if and only if $w_{a_i} < w_{a_j}$. Let $\mathcal{S}_n(u,v)$ denote the set of permutations $w \in \mathcal{S}_n$ avoiding both the permutations $u, v \in \mathcal{S}_4$. There is a group $G$ of order 16 that acts on the set of pairs $(u, v)$ of unequal elements of $\mathcal{S}_4$ such that if $(u, v)$ and $(u', v')$ are in the same $G$-orbit (in which case we say that they are equivalent), then there is a simple bijection between $\mathcal{S}_n(u, v)$ and $\mathcal{S}_n(u', v')$ (for all $n$). Namely, identifying a permutation with the corresponding permutation matrix, the orbit of $(u, v)$ is obtained by possibly interchanging $u$ and $v$, and then doing a simultaneous dihedral symmetry of the square matrices $u$ and $v$. There are then ten inequivalent pairs $(u, v) \in \mathcal{S}_4 \times \mathcal{S}_4$ for which $\# \mathcal{S}_n(u, v) = r_{n-1}$, namely, $(1234, 1243)$, $(1243, 1234)$, $(1243, 1342)$, $(1243, 2143)$, $(1324, 1342)$, $(1342, 1324)$, $(1342, 1432)$, $(1342, 2341)$, $(1342, 3142)$, and $(2413, 3142)$.

(m) $r_{n-1}$ is the number of permutations $w = w_1w_2 \cdots w_n$ of $[n]$ with the following property: It is possible to insert the numbers $w_1, \ldots, w_n$ in order into a string, and to remove the numbers from the string in the order $1, 2, \ldots, n$. Each insertion must be at the beginning or end of the string. At any time we may remove the first (leftmost) element of the string. (Example: $w = 2413$. Insert 2, insert 4 at the right, insert 1 at the left, remove 1, remove 2, insert 3 at the left, remove 3, remove 4.)

(n) $r_n$ is the number of sequences of length $2n$ from the alphabet $A, B, C$ such that:

(i) for every $1 \leq i < 2n$, the number of $A$’s and $B$’s among the first $i$ terms is not
Figure 9: A board with \( r_3 = 22 \) domino tilings

less than the number of \( C \)'s, (ii) the total number of \( A \)'s and \( B \)'s is \( n \) (and hence the also the total number of \( C \)'s), and (iii) no two consecutive terms are of the form \( CB \).

(o) \( r_{n-1} \) is the number of noncrossing partitions (as defined in Exercise 3.68) of some set \( |k| \) into \( n \) blocks, such that no block contains two consecutive integers.

(p) \( s_n \) is the number of graphs \( G \) (without loops and multiple edges) on the vertex set \( [n+2] \) with the following two properties: (α) All of the edges \( \{1, n+2\} \) and \( \{i, i+1\} \) are edges of \( G \), and (β) \( G \) is noncrossing, i.e., there are not both edges \( \{a, c\} \) and \( \{b, d\} \) with \( a < b < c < d \). Note that an arbitrary noncrossing graph on \( [n+2] \) can be obtained from those satisfying (α)-(β) by deleting any subset of the required edges in (α). Hence the total number of noncrossing graphs on \( [n+2] \) is \( 2^{n+2} s_n \).

(q) \( r_{n-1} \) is the number of reflexive and symmetric relations \( R \) on the set \( [n] \) such that if \( iRj \) with \( i < j \), then we never have \( uRv \) for \( i < u < j < v \).

(r) \( r_{n-1} \) is the number of reflexive and symmetric relations \( R \) on the set \( [n] \) such that if \( iRj \) with \( i < j \), then we never have \( uRv \) for \( i < u < j < v \).

(s) \( r_{n-1} \) is the number of ways to cover with disjoint dominos (or dimers) the set of squares consisting of \( 2i \) squares in the \( i \)th row for \( 1 \leq i \leq n - 1 \), and with \( 2(n-1) \) squares in the \( n \)th row, such that the row centers lie on a vertical line. See Figure 9 for the case \( n = 4 \).

40. [3] Let \( a_n \) be the number of permutations \( w = w_1 w_2 \cdots w_n \in \mathcal{S}_n \) such that we never have \( w_{i+1} = w_i \pm 1 \), e.g., \( a_4 = 2 \), corresponding to 2413 and 3142. Equivalently, \( a_n \) is the number of ways to place \( n \) nonattacking kings on an \( n \times n \) chessboard with one king in every row and column. Let

\[
A(x) = \sum_{n\geq 0} a_n x^n = 1 + x + 2x^4 + 14x^5 + 90x^6 + 646x^7 + 5242x^8 + \cdots.
\]

Show that \( A(x R(x)) = \sum_{n\geq 0} n! x^n := E(x) \), where

\[
R(x) = \sum_{n\geq 0} r_n x^n = \frac{1}{2x} \left( 1 - x - \sqrt{1 - 6x + x^2} \right),
\]

243
the generating function for Schröder numbers. Deduce that
\[ A(x) = E \left( \frac{x(1 - x)}{1 + x} \right). \]

41. [3] A permutation \( w \in \mathfrak{S}_n \) is called 2-stack sortable if \( S^2(w) = w \), where \( S \) is the operator of Exercise 19(ii). Show that the number \( S_2(n) \) of 2-stack sortable permutations in \( \mathfrak{S}_n \) is given by
\[ S_2(n) = \frac{2(3n)!}{(n + 1)! (2n + 1)!}. \]

42. [2] A king moves on the vertices of the infinite chessboard \( \mathbb{Z} \times \mathbb{Z} \) by stepping from \((i, j)\) to any of the eight surrounding vertices. Let \( f(n) \) be the number of ways in which a king can walk from \((0, 0)\) to \((n, 0)\) in \( n \) steps. Find \( F(x) = \sum_{n \geq 0} f(n)x^n \), and find a linear recurrence with polynomial coefficients satisfied by \( f(n) \).

43. (a) [2+] A secondary structure is a graph (without loops or multiple edges) on the vertex set \([n]\) such that (a) \( \{i, i + 1\} \) is an edge for all \( 1 \leq i \leq n - 1 \), (b) for all \( i \), there is at most one \( j \) such that \( \{i, j\} \) is an edge and \( |j - i| \neq 1 \), and (c) if \( \{i, j\} \) and \( \{k, l\} \) are edges with \( i < k < j \), then \( i \leq l \leq j \). (Equivalently, a secondary structure may be regarded as a 3412-avoiding involution (as in Exercise 19(kk)) such that no orbit consists of two consecutive integers.) Let \( s(n) \) be the number of secondary structures with \( n \) vertices. For instance, \( s(5) = 8 \), given by

\[ \begin{array}{c}
\quad \quad \quad \quad \quad \quad \quad \\
\quad \quad \quad \quad \quad \quad \quad \\
\quad \quad \quad \quad \quad \quad \quad \\
\quad \quad \quad \quad \quad \quad \quad \\
\quad \quad \quad \quad \quad \quad \quad \\
\quad \quad \quad \quad \quad \quad \quad \\
\end{array} \]

Let \( S(x) = \sum_{n \geq 0} s(n)x^n = 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + 37x^7 + 82x^8 + 185x^9 + 423x^{10} + \cdots \). Show that
\[ S(x) = \frac{x^2 - x + 1 - \sqrt{1 - 2x - x^2 + 2x^3 + x^4}}{2x^2}. \]

(b) [3−] Show that \( s(n) \) is the number of walks in \( n \) steps from \((0, 0)\) to the \( x \)-axis, with steps \((1, 0)\), \((0, 1)\), and \((0, -1)\), never passing below the \( x \)-axis, such that \((0, 1)\) is never followed directly by \((0, -1)\).

44. Define a Catalan triangulation of the Möbius band to be an abstract simplicial complex triangulating the Möbius band that uses no interior vertices, and has vertices labelled \( 1, 2, \ldots, n \) in order as one traverses the boundary. (If we replace the Möbius band by a disk, then we get the triangulations of Corollary 2.3(vi) or Exercise 19(a).) Figure 10 shows the smallest such triangulation, with five vertices (where we identify the vertical edges of the rectangle in opposite directions). Let \( MB(n) \) be the number of Catalan
triangulations of the Möbius band with \( n \) vertices. Show that

\[
\sum_{n \geq 0} MB(n) x^n = \frac{x^2 \left( (2 - 5x - 4x^2) + (-2 + x + 2x^2)\sqrt{1 - 4x} \right)}{(1 - 4x)(1 - 4x + 2x^2 + (1 - 2x)\sqrt{1 - 4x})} = x^5 + 14x^6 + 113x^7 + 720x^8 + 4033x^9 + 20864x^{10} + \cdots.
\]

45. [3-] Let \( f(n) \) be the number of nonisomorphic \( n \)-element posets with no 3-element antichain. For instance, \( f(4) = 10 \), corresponding to

\[
\begin{align*}
\bullet & \quad \bullet & & \bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & & \bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & & \bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & & \bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
\]

Let \( F(x) = \sum_{n \geq 0} f(n) x^n = 1 + x + 2x^2 + 4x^3 + 10x^4 + 26x^5 + 75x^6 + 225x^7 + 711x^8 + 2311x^9 + 7725x^{10} + \cdots \). Show that

\[
F(x) = \frac{4}{2 - 2x + \sqrt{1 - 4x} + \sqrt{1 - 4x^2}}.
\]

46. (a) [3+] Let \( f(n) \) denote the number of subsets \( S \) of \( \mathbb{N} \times \mathbb{N} \) of cardinality \( n \) with the following property: If \( p \in S \) then there is a lattice path from \((0,0)\) to \( p \) with steps \((0,1)\) and \((1,0)\), all of whose vertices lie in \( S \). Show that

\[
\sum_{n \geq 1} f(n) x^n = \frac{1}{2} \left( \sqrt{\frac{1 + x}{1 - 3x}} - 1 \right) = x + 2x^2 + 5x^3 + 13x^4 + 35x^5 + 96x^6 + 267x^7 + 750x^8 + 2123x^9 + 6046x^{10} + \cdots.
\]

(b) [3+] Show that the number of such subsets contained in the first octant \( 0 \leq x \leq y \) is the Motzkin number \( M_{n-1} \) (defined in Exercise 37).

47. (a) [3] Let \( P_n \) be the Bruhat order on the symmetric group \( \mathfrak{S}_n \) as defined in Exercise 3.75(a). Show that the following two conditions on a permutation \( w \in \mathfrak{S}_n \) are equivalent:

i. The interval \([\bar{0}, w]\) of \( P_n \) is rank-symmetric, i.e., if \( \rho \) is the rank function of \( P_n \) (so \( \rho(w) \) is the number of inversions of \( w \)), then

\[
\# \{ u \in [\bar{0}, w] : \rho(u) = i \} = \# \{ u \in [\bar{0}, w] : \rho(w) - \rho(u) = i \},
\]

for all \( 0 \leq i \leq \rho(w) \).
ii. The permutation \( w = w_1w_2 \cdots w_n \) is 4231 and 3412-avoiding, i.e., there do not exist \( a < b < c < d \) such that \( w_d < w_b < w_c < w_a \) or \( w_c < w_d < w_a < w_b \).

(b) [3–] Call a permutation \( w \in \mathfrak{S}_n \) smooth if it satisfies (i) (or (ii)) above. Let \( f(n) \) be the number of smooth \( w \in \mathfrak{S}_n \). Show that

\[
\sum_{n \geq 0} f(n) x^n = \frac{1}{1 - x - \frac{x^2}{1-x} \left( \frac{2x}{1+x-(1-x)C(x)} - 1 \right)}
\]

\[= 1 + x + 2x^2 + 6x^3 + 22x^4 + 88x^5 + 366x^6 + 1552x^7 + 6652x^8 + 28696x^9 + \cdots,
\]

where \( C(x) = (1 - \sqrt{1 - 4x})/2x \) is the generating function for the Catalan numbers.

48. [3] Let \( f(n) \) be the number of 1342-avoiding permutations \( w = w_1w_2 \cdots w_n \) in \( \mathfrak{S}_n \), i.e., there do not exist \( a < b < c < d \) such that \( w_a < w_d < w_b < w_c \). Show that

\[
\sum_{n \geq 0} f(n) x^n = \frac{32x}{1 + 20x - 8x^2 - (1 - 8x)^{3/2}}
\]

\[= 1 + x + 2x^2 + 6x^3 + 23x^4 + 103x^5 + 312x^6 + 740x^7 + 15485x^8 + \cdots.
\]

49. (a) [3–] Let \( B_n \) denote the board consisting of the following number of squares in each row (read top to bottom), with the centers of the rows lying on a vertical line: 2, 4, 6, \ldots, 2(n – 1), 2n (three times), 2(n – 1), \ldots, 6, 4, 2. Figure 11 shows the board \( B_3 \). Let \( f(n) \) be the number of ways to cover \( B_n \) with disjoint dominoes (or dimers). (A domino consists of two squares with an edge in common.) Show that \( f(n) \) is equal to the central Delannoy number \( D(n, n) \) (as defined in Section 3).

(b) [3–] What happens when there are only two rows of length \( 2n \)?

50. [3] Let \( B \) denote the “chessboard” \( \mathbb{N} \times \mathbb{N} \). A position consists of a finite subset \( S \) of \( B \), whose elements we regard as pebbles. A move consists of replacing some pebble, say at cell \( (i, j) \), with two pebbles at cells \( (i+1, j) \) and \( (i, j+1) \), provided that each of these cells is not already occupied. A position \( S \) is reachable if there is some sequence of moves, beginning with a single pebble at the cell \( (0, 0) \), that terminates in the position \( S \). A subset \( T \) of \( B \) is unavoidable if every reachable set intersects \( T \). A subset \( T \) of \( B \) is minimally unavoidable if \( T \) is unavoidable, but no proper subset of \( T \) is unavoidable. Let \( u(n) \) be the number of \( n \)-element minimally unavoidable subsets of \( B \). Show that

\[
\sum_{n \geq 0} u(n) x^n = x^3 \frac{(1 - 3x + x^2) \sqrt{1 - 4x} - 1}{1 - 7x + 14x^2 - 9x^3}
\]

\[= 4x^5 + 22x^6 + 98x^7 + 412x^8 + 1700x^9 + 6974x^{10} + 28576x^{11} + \cdots.
\]
Figure 11: A board with $D(3, 3) = 63$ domino tilings