Exercise 3.1. Think again about the linear program derived from the diet problem in Exercise 2.4 (Homework 2).

(a) Starting with the previously obtained non-optimal vertex $x_0$ from part (c) of Exercise 2.4, execute simplex steps to find the optimal solution $x^*$. (Only a few steps will be needed.) At each iteration $k$ of the simplex method, please give $x_k$, $\lambda_k$, $p_k$, $\alpha_k$, and $c^T x_k$.

(i) Give the optimal solution $x^*$, the active set at $x^*$, and the associated Lagrange multipliers.

(ii) Explain how you know that $x^*$ is optimal.

(iii) Comment on any notable features of $x^*$, and also on your reaction to the diet given by $x^*$.

(b) Now we will solve a related but different problem. Start again with the non-optimal vertex $x_0$ from part (c) of Exercise 2.4, but this time execute simplex steps to find the least enjoyable diet $\tilde{x}$ that satisfies the same constraints, giving all relevant information at each iteration.

(i) Give the least-enjoyable diet $\tilde{x}$, the active set at $\tilde{x}$, and the associated Lagrange multipliers.

(ii) Explain how the problem formulation is different from the case when enjoyment is maximized.

(iii) Explain how you know that your solution is optimal for this (reversed) problem.

(iv) Comment on any interesting differences from the solution that you found in part (a).

Exercise 3.2. Let following inconsistent linear inequalities $Ax \geq b$, containing 3 constraints in 2 variables, be defined as the “original” constraints in this problem:

\[
A = \begin{pmatrix}
4 & -1 \\
-1 & 0 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix}.
\]

(3.1)

(a) The following all-inequality phase-1 linear program is associated with the original constraints in (3.1): minimize $\tilde{c}^T \tilde{x}$ subject to $\tilde{A} \tilde{x} \geq \tilde{b}$, where the matrix $\tilde{A}$ and the vectors $\tilde{b}$ and $\tilde{c}$ are given by

\[
\tilde{A} = \begin{pmatrix}
4 & -1 & 1 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix}, \quad \text{and} \quad \tilde{c} = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]

(3.2)

(i) Explain exactly how a linear program with $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ given in (3.2) is related to trying to decrease infeasibility for the constraints of the original problem. What is the significance of an optimal solution of the all-inequality LP defined by (3.2)?

(ii) Explain why $\tilde{x} = (1, 2, 1)^T$ is an optimal vertex for the phase-1 LP specified by $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ of (3.2). What does $\tilde{x}$ tell us about the original constraints (3.1)?
(b) Given a generic set of inequality constraints \( Ax \geq b \), where \( A \) is \( m \times n \), consider the phase-1 LP of minimizing \( \hat{c}^T \hat{x} \) subject to \( \hat{A} \hat{x} \geq \hat{b} \), with

\[
\hat{A} = \left( \begin{array}{c} A \\ I_m \\ I_m \end{array} \right), \quad \hat{b} = \left( \begin{array}{c} b \\ 0_m \\ 0_m \end{array} \right), \quad \text{and} \quad \hat{c} = \left( \begin{array}{c} 0_n \\ e \end{array} \right),
\]

where \( \hat{A} \) is \( 2m \times (m+n) \), \( I_m \) is the \( m \)-dimensional identity matrix, \( 0_{m \times n} \) means an \( m \times n \) block of zeros, \( 0_m \) means a column of \( m \) zeros, and \( e \) is vector of compatible dimension whose components are all equal to 1, \( e = (1, 1, \ldots, 1)^T \). How exactly is the solution of this LP related to the original constraints?

(c) Consider the phase-1 LP defined by (3.3), where the entries of \( A \) and \( b \) are given in (3.1). Explain how to use this LP to show that the smallest sum of infeasibilities for the original constraints is achieved when \( x = (1.25, 2) \), where \( x \) denotes the original variables.

**Exercise 3.3.** Here is an optimization problem with a linear objective function and linear inequality constraints:

\[
\begin{align*}
\text{minimize} & \quad -x_1 - 2x_2 + 2x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + x_3 \leq 4 \\
& \quad -x_1 + x_2 + x_3 \leq 0 \\
& \quad 2x_2 + x_3 \leq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \geq 0
\end{align*}
\]

(a) Convert this problem into a standard-form linear program, being sure to note the direction of the inequalities in the first three constraints. Give (explicitly) the matrix \( A \) in the resulting standard-form problem, describe all the variables, and explain how the conversion was done.

(b) Perform one iteration of the standard-form simplex method, starting with the basic set \( \{4, 1, 3\} \) (so that the basis matrix \( B \) will consist of columns 4, 1, and 3 in the matrix \( A \) from your standard-form problem), with the three basic variables \( x_B \) all equal to 1. Show all relevant calculations, including the Lagrange multipliers \( \pi \) and \( z \), the entering variable, the search direction for the basic variables, the maximum feasible step, and the leaving variable. Confirm that the new iterate is feasible, with improved objective value.

**Exercise 3.4.** Consider the following standard-form linear program with constraints \( Ax = b \) and \( x \geq 0 \), and the specific point \( \bar{x} \):

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \text{and} \quad \bar{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\]

Is \( \bar{x} \) a vertex with respect to the standard-form constraints? Explain why or why not.

**Exercise 3.5.** This problem involves the famous Klee-Minty LP\(^1\) that produces worst-case behavior by the simplex method. The Klee-Minty LP can be described in numerous rescaled and rearranged forms, two of which appear in this problem.

(a) In one 2-d version of Klee-Minty, the problem is to maximize \( 10x_1 + x_2 \) subject to \( 0 \leq x_1 \leq 1, \ x_2 \geq 0, \) and \( 20x_1 + x_2 \leq 100 \).

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(i) How would you formulate this problem as an all-inequality linear program? Give $A$, $b$, and $c$.
(ii) Starting with the vertex $x_0 = (0, 0)$, execute steps of the simplex method using the textbook rule until you obtain the optimal vertex. Be sure to include your calculations.
(iii) What is the optimal vertex?
(iv) How many vertices were visited, including the initial vertex?

(b) Now consider an alternative (and equivalent) formulation of Klee–Minty in which a value $\epsilon$ is given such that $0 < \epsilon < \frac{1}{2}$. In $n$ dimensions, the constraints consist of the following pairs of upper and lower bounds,

$$
\epsilon \leq x_1 \leq 1, \quad \epsilon x_{j-1} \leq x_j \leq 1 - \epsilon x_{j-1}, \quad j = 2, \ldots, n,
$$

so that each successive variable is bounded above and below in terms of the preceding variable. Show that there are $2^n$ nondegenerate vertices for these constraints.