Whenever calculations are needed to solve a problem, those calculations (and, usually, your code) must be submitted as part of the homework assignment.

Homework must be submitted electronically. Unless express permission has been given in advance by the instructor for a late homework submission, a 30% percent penalty will be deducted for each late day (or part of a late day).

In printing non-integers, be sure to use scientific notation and to show at least 6 decimal digits following the decimal point.

Exercise 2.1. Let $A$ be a nonzero $m \times n$ matrix.

1. If $b$ is an $m$-vector such that $b_i \leq 0$ for $i = 1, \ldots, m$, show that there is at least one point that satisfies the combined constraints $Ax \geq b$ and $x \geq 0$.

2. If there are no restrictions on the signs of the components of $b$, can it still be guaranteed that a feasible point exists? Explain why or why not. In the latter case, give a (small) counterexample.

Exercise 2.2. Consider the linear equality constraints $Ax = b$ and the vector $\tilde{x}$:

$$A = \begin{pmatrix} 3 & -4 & 6 & -1 & 7 \\ 2 & 2 & -3 & 4 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -40 \\ 7 \end{pmatrix}, \text{ and } \tilde{x} = (-3, 3, -2, 0, -1)^T.$$

Let $c = (-11, -4, 6, -15, -3)^T$ and $d = (1, 1, 1, 1, 1)^T$.

(a) Show (using computation) that $\tilde{x}$ is feasible.

(b) Find any feasible point $\bar{x}$ such that $\bar{x} \neq \tilde{x}$ and explain how you found it.

(c) Consider the problem of minimizing the objective function $\ell(x) = c^T x$ subject to $Ax = b$.

(i) Is $\tilde{x}$ optimal for this problem?

(ii) If “yes”, explain why. In this case, also explain whether $\tilde{x}$ is the unique minimizer, and whether or not the optimal value of the objective function is unique.

(iii) If $\tilde{x}$ is not optimal, explain why it is not. In this case, find a direction $p$ such that $Ap = 0$ and $c^T p < 0$ (and explain how you found $p$).

(d) Now consider the problem of minimizing $\ell(x) = d^T x$ subject to $Ax = b$.

(i) Is $\tilde{x}$ optimal for this problem?

(ii) If “yes”, explain why. In this case, also explain whether $\tilde{x}$ is the unique minimizer, and whether or not the optimal value of the objective function is unique.

(iii) If $\tilde{x}$ is not optimal, explain why it is not, find a direction $p$ such that $Ap = 0$ and $d^T p < 0$, and be sure to explain how you found $p$.

Exercise 2.3. A constraint in a set of inequalities is called redundant if its removal does not alter the feasible region. For example, given the constraints $x_1 \geq 2$ and $x_1 \geq 4$, the first is redundant. Detecting redundant constraints is difficult in general, but is sometimes straightforward, as in this problem.
(a) Consider the feasible region defined by the three constraints \( x_1 \geq 1, \ x_2 \geq 1 \) and \( x_1 + x_2 \geq 2 \). Which constraint (if any) is redundant? Justify your answer.

(b) Consider the feasible region defined by the three constraints \( x_1 \geq 1, \ x_2 \geq 1 \) and \( x_1 + x_2 \leq 2 \). (Note the change of direction in the third constraint compared to part (a).) Describe the feasible region. Which constraint (if any) is redundant? Justify your answer.

**Exercise 2.4.** Suppose that four foods are available to you: milk, chocolate chip cookies, kale, and pizza. Your daily diet right now consists of 2 pints of milk, 30 cookies, \( \frac{1}{2} \) bowl of kale, and 5 slices of pizza. You have recently been informed by an authoritative source that you must consume more fiber and less fat, without adding any new foods to your diet.

You therefore wish to determine nonnegative quantities of the four foods available to you to make up a diet that maximizes your personal enjoyment while satisfying these conditions: (1) your daily fat intake must be less than or equal to 3,000; (2) you must consume at least 65 units of vitamin Z and 25 units of vitamin Y each day; and (3) you must consume at least 700 units of fiber each day.

The following table shows the amounts of vitamins Z and Y, fiber, fat, and enjoyment per indicated quantity of the four foods.

<table>
<thead>
<tr>
<th>Food</th>
<th>Z</th>
<th>Y</th>
<th>Fiber</th>
<th>Fat</th>
<th>Enjoyment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milk (pint)</td>
<td>55</td>
<td>12</td>
<td>7</td>
<td>78</td>
<td>175</td>
</tr>
<tr>
<td>Cookies (each)</td>
<td>2</td>
<td>14</td>
<td>12</td>
<td>240</td>
<td>225</td>
</tr>
<tr>
<td>Kale (bowl)</td>
<td>15</td>
<td>32</td>
<td>210</td>
<td>60</td>
<td>-200</td>
</tr>
<tr>
<td>Pizza (slice)</td>
<td>34</td>
<td>45</td>
<td>7</td>
<td>800</td>
<td>450</td>
</tr>
</tbody>
</table>

(a) Formulate a linear program of the form “minimize \( c^T x \) subject to \( Ax \geq b \)” whose solution will give you the most enjoyable diet satisfying conditions (1), (2), and (3), i.e., give \( A, b, \) and \( c \).

(b) Is your current diet feasible? Explain.

(c) Find a non-optimal vertex \( x_0 \), explain how you found it, and compute \( c^T x_0 \). Explain how you know that \( x_0 \) is a vertex, and how you know that \( x_0 \) is not optimal.

**Exercise 2.5.** Consider the linear program of minimizing \( c^T x \) subject to \( Ax \geq b \), where \( A \) is \( m \times n \). Let \( x \) be a nondegenerate vertex where the active set is \( \mathcal{A} \) and the active-constraint matrix is \( A_{\mathcal{A}} \). Assume that

\[
c = A_{\mathcal{A}}^T \lambda \quad \text{and} \quad \lambda \geq 0,
\]

but \( \lambda_i = 0 \) for one or more indices \( i \). Under these conditions, prove that \( x \) is not the unique optimal solution of the linear program.

**Exercise 2.6.** The following LP was constructed by Harold Kuhn (the second “K” in the KKT conditions) to illustrate that use of the textbook rules can lead to cycling:

\[
\begin{align*}
\text{minimize} & \quad -2x_1 - 3x_2 + x_3 + 12x_4 \\
\text{subject to} & \quad x_1, \ x_2, \ x_3, \ x_4 \geq 0 \\
& \quad 2x_1 + 9x_2 - x_3 - 9x_4 \geq 0 \\
& \quad -\frac{1}{3}x_1 - 2x_2 + \frac{1}{3}x_3 + 2x_4 \geq 0.
\end{align*}
\]

(a) Run the simplex method on this problem using the textbook rules, starting with \( x_0 \) as the origin (a degenerate vertex where all six constraints are active) and taking the initial working set as \( \mathcal{W}_0 = \{1, 2, 3, 4, \} \). How many iterations occur before the initial working set is repeated (possibly in reordered form)?

(b) Now apply the simplex method, with the same initial point and working set, using Bland’s least-index choice rules, and demonstrate that the iterates move away from the origin at a subsequent iteration.

(c) Does the Kuhn LP have a bounded solution? Explain.