YOUR NAME:

Final, December 22, 2011

Numerical Methods I

Cross out what is not meant to be part of your final answer.

1. A penta-diagonal matrix A is a square matrix for which all elements $a_{ij} = 0, \; for \; all \; |i - j| > 2.$

   (a) Show that we can solve a linear system $Ax = b$ with a penta-diagonal A in order n operations and using only a fixed number of vectors of length n for storage. Discuss both the case when no pivoting is needed and the case when pivoting is required.

   (b) Consider the case of a symmetric, positive definite, penta-diagonal matrix A. Cholesky's method can then be applied. Is it backward stable? What can we say about the accuracy of the solution?

2. Simpson's rule

$$\frac{b-a}{6}(f(a) + 4f((a+b)/2) + f(b))$$

   provides an approximate value of $\int_a^b f(x)dx$.

   (a) The method is designed to be exact for all polynomials of degree 2. Explain, concisely, how this is done.

   (b) Show that Simpson's rule is also exact for all cubic polynomials. To simplify the computation, make $a = -b$.

   (c) For sufficiently smooth functions $f(x)$ it is known that the error equals $-(b-a)^4/2880 f^{(4)}(\xi)$ for some $\xi \in (a, b)$. Can you use this result to prove that Simpson's rule is exact for all cubic polynomials?

   (d) Briefly outline how this formula forms the basis for the adaptive Simpson method.

3. The Chebyshev polynomials $T_n(x), n = 0, 1, \ldots$ can be defined by $T_n(x) = \cos(n \arccos(x))$ for $-1 \le x \le 1$.

   (a) Use the trigonometric identity $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(\theta)\cos(n\theta)$ to derive a three-term recurrence for these polynomials. Use the original formula to obtain $T_0(x)$ and $T_1(x)$.

(b) With respect to which inner product are these polynomials an orthogonal basis for the space of polynomials?

(c) Derive the two point Gaussian quadrature formula $A_0 f(x_0) + A_1 f(x_1)$ using the Chebyshev polynomials. Determine the coefficients A_0 and A_1 and the values of x_0 and x_1. Assume that we work on the interval $[-1, 1]$.

(d) For which family of integrands does this formula give the exact answer?

4. (a) What is Runge's phenomenon? (Recall that it is about polynomial interpolation of $f(x) = \frac{1}{1+25x^2}$, $x \in [-1, 1]$.)

   (b) Explain what happens when we use the zeros of the Chebyshev polynomial T_{n+1} as the point set where we interpolate. Give an explanation why this is a better choice than using equidistant points.

   (c) Explain, concisely, how to set up a least squares problem using the data at $n + 1$ equidistant points and the $f(x)$ given above to compute the best quadratic polynomial in the sense of least squares.

5. (a) What is a Hessenberg matrix?

   (b) What is a Householder transformation?

   (c) Explain how to use on the order n Householder transformations to transform a $n \times n$ matrix A into a Hessenberg matrix H with the same set of eigenvalues as A.

6. Let us solve a large linear system of equations $Ax = b$ where A is a symmetric, positive definite square matrix. This is first done by Richardson's method: Given an initial guess x_0, the iteration is advanced by

$$x_k = x_{k-1} - \alpha(Ax_{k-1} - b), k = 1, 2, \ldots.$$

   Here α is a positive parameter.

   (a) Suppose that we observe, by running a number of experiments, that the rate of convergence is at least as good as 0.6. What can we then conclude about the condition number of A?

   (b) Suppose we now switch to the conjugate gradient method. What can we say about its convergence rate given the information obtained from the experiments using the simpler iterative algorithm?