Image Formation

Light can change the image and appearances (images from D. Jacobs)
What is the relation between pixel brightness and scene radiance?
Later: what is the relation between pixel brightness and scene reflectance?
Image Formation

\[ \delta \Omega = \frac{\delta A \cos \theta}{R^2} \]

equals solid angle subtended by a small patch of area \( A \).

\[ \hat{n} \quad \delta \Omega \quad R \]

\[ \delta A \]

\( \theta \)

L - radiance is the amount of light radiated from a surface per solid angle (power per unit area per unit solid angle emitted from a surface. \( W \, m^{-2} \, sr^{-1} \))

E - irradiance is the amount of light falling in a surface (power per unit area incident in a surface. \( W \, m^{-2} \))
Surface Radiance and Image Irradiance

Pinhole Camera Model

\[ \frac{\delta A \cos \theta}{(z / \cos \alpha)^2} = \frac{\delta I \cos \alpha}{(f / \cos \alpha)^2} \rightarrow \frac{\delta A}{\delta I} = \frac{\cos \alpha}{\cos \theta} \left( \frac{z}{f} \right)^2 \]
Surface Radiance and Image Irradiance

Solid angle subtended by the lens, as seen by the patch $\delta A$

$$\Omega = \frac{\pi}{4} \frac{d^2 \cos \alpha}{(z / \cos \alpha)^2} = \frac{\pi}{4} \left( \frac{d}{z} \right)^2 \cos^3 \alpha$$

Power from patch $\delta A$ through the lens

$$\delta P = L \Omega \delta A \cos \theta = L \delta A \frac{\pi}{4} \left( \frac{d}{z} \right) \cos^3 \alpha \cos \theta$$

Thus, we conclude

$$E = \frac{\delta P}{\delta I} = L \frac{\delta A \pi}{\delta I} \left( \frac{d}{z} \right)^2 \cos^3 \alpha \cos \theta = L \frac{\pi}{4} \left( \frac{d}{f} \right)^2 \cos^4 \alpha$$
Conclusions

\[ E = L \frac{\pi}{4} \left( \frac{d}{f} \right)^2 \cos^4 \alpha \]

• The irradiance at the image pixel is converted into the brightness of the pixel

• Image Irradiance is proportional to Scene Radiance

• Scene distance, z, does not affect/reduce image brightness (the model is too simplified, since in practice it does.)

• The angle of the scene patch with respect to the view (\( \alpha \)) reduces the brightness by the \( \cos^4 \alpha \). In practice the effect is even stronger.
The Bidirectional Reflectance Distribution Function (BRDF)

\[ f(\theta_i, \phi_i, \theta_e, \phi_e) = \frac{\delta L(\theta_e, \phi_e)}{\delta E(\theta_i, \phi_i)} \]

BRDF - How bright a surface appears when viewed from one direction while light falls on it from another.

Usually \( f \) depends only on \( \phi_e - \phi_i, \theta_i, \theta_e \): true for matte surfaces and specularly reflecting surfaces.
Extended Light Sources and BRDF

\[ \delta \Omega = \sin \theta_i \delta \theta_i \delta \phi_i \]

Light source radiance arriving through solid angle \( \delta \Omega \)

\[ \delta E (\theta_i, \phi_i) = E(\theta_i, \phi_i) \sin \theta_i \delta \theta_i \delta \phi_i \]

Power arriving at patch \( \delta A \) from \( \delta \Omega \)

\[ \delta P = \delta A \cos \theta_i \delta E (\theta_i, \phi_i) = \delta A \ E(\theta_i, \phi_i) \cos \theta_i \sin \theta_i \delta \theta_i \delta \phi_i \]

Foreshortening

thus the irradiance arriving at patch \( \delta A \) is

\[ E_0 = \frac{\int \delta P}{\delta A} = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} E(\theta_i, \phi_i) \cos \theta_i \sin \theta_i \delta \theta_i \delta \phi_i \]

The radiance of a patch \( \delta A \) at direction \((\theta_e, \phi_e)\) is thus, given by

\[ L(\theta_e, \phi_e) = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} f(\theta_i, \phi_i, \theta_e, \phi_e) \ E(\theta_i, \phi_i) \cos \theta_i \sin \theta_i \delta \theta_i \delta \phi_i \]
Special Cases of BRDF

1. **Lambertian Surfaces (matte)** - appears equally bright from all viewing directions and reflects all incident light, absorbing none, i.e. the BRDF is constant and \( L = E_0 \). What constant \( f \)?

\[
L(\theta_e, \phi_e) = f \int_{-\pi}^{\pi} \int_{0}^{\frac{\pi}{2}} E(\theta_i, \phi_i) \cos \theta_i \sin \theta_i \delta \theta_i \delta \phi_i = f \ E_0
\]

Thus, the total "reflected power" from patch \( \delta A \) becomes

\[
\delta p = \int_{-\pi}^{\pi} \int_{0}^{\frac{\pi}{2}} L(\theta_e, \phi_e) \delta A \cos \theta_e \sin \theta_e \delta \theta_e \delta \phi_e = f \ E_0 \delta A \pi
\]

since

\[
\int_{-\pi}^{\pi} \int_{0}^{\frac{\pi}{2}} \cos \theta_e \sin \theta_e \delta \theta_e \delta \phi_e = \pi
\]

Using that \( L = \frac{\delta p}{\delta A} = f \ E_0 \pi \) and for Lambertian surfaces \( L = E_0 \),

we finally obtain

\[
f = \frac{1}{\pi}
\]
Special Cases of BRDF

2. **Specular Surfaces (mirrors)** – reflects all light arriving from the direction \((\theta_i, \phi_i)\) into the direction \((\theta_e, \phi_i + \pi)\). The BRDF is in this case proportional to the product of two impulses, \(\delta(\theta_e - \theta_i)\) and \(\delta(\phi_e - \phi_i - \pi)\). What is the factor of proportionality \(k(\theta_i, \phi_i)\)?

\[
L(\theta_e, \phi_e) = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} k(\theta_i, \phi_i) \delta(\theta_e - \theta_i) \delta(\phi_e - \phi_i - \pi) E(\theta_i, \phi_i) \cos \theta_i \sin \theta_i \delta \theta_i \delta \phi_i \\
= k(\theta_e, \phi_e) E(\theta_e, \phi_e) \cos \theta_e \sin \theta_e
\]

\[
E_0 = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} E(\theta_i, \phi_i) \cos \theta_i \sin \theta_i \delta \theta_i \delta \phi_i
\]

\[
L = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} L(\theta_e, \phi_e) \cos \theta_e \sin \theta_e \delta \theta_e \delta \phi_e
\]

And for specular surfaces \(L = E_0 \Rightarrow k(\theta_i, \phi_i) = \frac{1}{\cos \theta_i \sin \theta_i}\)

we finally obtain \(f(\theta_i, \phi_i, \theta_e, \phi_e) = \frac{\delta(\theta_e - \theta_i) \delta(\phi_e - \phi_i - \pi)}{\cos \theta_i \sin \theta_i}\)
Lambertian Surface Brightness

How bright will a Lambertian surface be when it is illuminated by a point source of radiance $E$? and by a “sky” of uniform radiance $E$?

For a point source the irradiance at the surface is $E_0 = E \cos \theta_i$ and the radiance must then be

$$L(\theta_e, \phi_e) = f \ E_0 = \frac{1}{\pi} E \cos \theta_i$$

Familiar cosine or “Lambert’s law” of reflection from matte surfaces (surfaces covered with finely powdered transparent materials such as barium sulfate or magnesium carbonate), and can approximate paper, snow and matte paint.

Finally, for a “sky” of uniform radiance $E$ we obtain

$$L(\theta_e, \phi_e) = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} \frac{1}{\pi} E \cos \theta_i \sin \theta_i \ \delta \theta_i \ \delta \phi_i = E!$$