The Sample Complexity of Revenue Maximization

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Abstract

In the design and analysis of revenue-maximizing auctions, auction performance is typically measured with respect to a prior distribution over inputs. The most obvious source for such a distribution is past data. The goal of this paper is to understand how much data is necessary and sufficient to guarantee near-optimal expected revenue.

Our basic model is a single-item auction in which bidders' valuations are drawn independently from unknown and non-identical distributions. The seller is given m samples from each of these distributions "for free" and chooses an auction to run on a fresh sample. How large does m need to be, as a function of the number k of bidders and $\epsilon > 0$, so that a $(1-\epsilon)$-approximation of the optimal revenue is achievable?

We prove that, under standard tail conditions on the underlying distributions, $m = \text{poly}(k, \frac{1}{\epsilon})$ samples are necessary and sufficient. Our lower bound stands in contrast to many recent results on simple and prior-independent auctions and fundamentally involves the interplay between bidder competition, non-identical distributions, and a very close (but still constant) approximation of the optimal revenue. It effectively shows that the only way to achieve a sufficiently good constant approximation of the optimal revenue is through a detailed understanding of bidders' valuation distributions. Our upper bound is constructive and applies in particular to a variant of the empirical Myerson auction, the natural auction that runs the revenue-maximizing auction with respect to the empirical distributions of the samples.

Our sample complexity lower bound depends on the set of allowable distributions, and to capture this we introduce α-strongly regular distributions, which interpolate between the well-studied classes of regular ($\alpha = 0$) and MHR ($\alpha = 1$) distributions. We give evidence that this definition is of independent interest.

1 Introduction

Comparing the revenue of two different auctions requires an analysis framework for trading off performance on different inputs. For instance, in a single-item auction, a second-price auction with a reserve price $r > 0$ will earn more revenue than a second-price auction with no reserve price on some inputs, and less on others. Which auction is better?

The conventional approach in auction theory is Bayesian, or average-case, analysis. That is, bidders' valuations are assumed to be drawn from a distribution, and one auction is defined to be better than another if it has higher expected revenue with respect to this distribution. The optimal auction is then the one with the highest expected revenue. The optimal auction depends on the assumed distribution, in some cases in a detailed way.

While there is now a significant body of work on worst-case revenue maximization (see [12]), a majority of modern computer science research on revenue-maximizing auctions uses Bayesian analysis to measure auction performance (see [11]). Since the comparison between auctions depends fundamentally on the assumed distribution, an obvious question is: *where does this prior distribution come from, anyway?*

In most applications, and especially in computer science contexts, the answer is equally obvious: *from past data*. For example, in Yahoo!'s keyword auctions, Bayesian analysis is used to provide guidance on how to set per-click reserve prices, and the valuation distributions used in this analysis are derived straightforwardly from bid data from the recent past [18]. This is a natural approach, but how well does it work?

1.1 The Model

The goal of this paper is to understand how much data is necessary and sufficient to guarantee near-optimal expected revenue. Our most basic model is the following. There are k bidders in a single-item auction. The valuation (i.e., willingness-to-pay) of bidder i is a sample from a distribution F_i. The F_i's are independent but not necessarily identical.

The distribution $\mathbf{F} = F_1 \times \cdots \times F_k$ is unknown to the seller. The "data" comes in the form of m i.i.d. samples $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m)}$ from $\mathbf{F}$ — equivalently, m i.i.d. samples from each of the k individual distributions $F_1, \ldots, F_k$. The seller observes the samples and then commits to an auction $\mathcal{A}$. We call this function from samples to auctions an *m-sample auction strategy*. The seller then earns the revenue of its chosen auction $\mathcal{A}$ on the "real" input, a fresh independent sample $\mathbf{v}^{(m+1)}$ from $\mathbf{F}$. We can state our main question as follows.

(*) *How many samples m are necessary and sufficient for the existence of an m-sample auction strategy that, for every distribution $\mathbf{F}$ in some class $\mathcal{D}$, has expected revenue at least $(1-\epsilon)$ times that of the optimal auction for $\mathbf{F}$?*

The expected revenue of an auction strategy is w.r.t. both the samples $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m)}$ and the input $\mathbf{v}^{(m+1)}$ — i.e., over $m+1$ i.i.d. samples from $\mathbf{F}$. The expected revenue of an optimal auction is w.r.t. a single sample (the input) from $\mathbf{F}$.

The answer to the question (*) could be a function of up to three different parameters: the error tolerance ϵ, the number k of bidders, and the set $\mathcal{D}$ of allowable distributions[1]. It is clear that some restriction on $\mathcal{D}$ is necessary for the question (*) to be interesting: without any restriction, no finite number of samples is sufficient to guarantee near-optimal revenue, even when there is only one bidder.[2]

[1] As the distribution $\mathbf{F}$ is unknown, we seek uniform sample complexity bounds, meaning bounds that depend only on $\mathcal{D}$ and not on $\mathbf{F}$.

[2] To see this, consider all distributions that take on a value M^2 with probability $\frac{1}{M}$ and 0 with probability

Two distributional assumptions that have been extensively used (see e.g. [11]) are the *regularity* and *monotone hazard rate (MHR)* conditions. The former asserts that the "virtual valuation" function $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ is nondecreasing, where f_i is the density of F_i, while the second imposes the strictly stronger condition that $\frac{f_i(v_i)}{1-F_i(v_i)}$ is nondecreasing. The "most tail-heavy" regular distribution has the distribution function $F_i(v_i) = 1 - \frac{1}{v_i+1}$, while the most tail-heavy MHR distributions are the exponential distributions.

Our lower bound on the sample complexity of revenue maximization depends on the set of allowable distributions, and to capture this we introduce a parameterized condition that interpolates between the regularity and MHR conditions; this condition is also useful in other contexts (see Section 4).

Definition 1.1. (α-Strongly Regular Distribution) *Let F be a distribution with positive density function f on its support $[a, b]$, where $0 \leq a < \infty$ and $a \leq b \leq \infty$. Let $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ denote the corresponding virtual valuation function. F is* α-strongly regular *if*

$$\varphi(y) - \varphi(x) \geq \alpha(y - x) \tag{1}$$

whenever $y > x \geq 0$.

For distributions with a differentiable virtual valuation function φ, condition (1) is equivalent to $\frac{d\varphi}{dv} \geq \alpha$. Regular and MHR distributions are precisely the 0- and 1-strongly regular distributions, respectively. A product distribution $\mathbf{F} = F_1 \times \cdots \times F_k$ is called α-strongly regular if each F_i is α-strongly regular. For the lower bound, we take the set $\mathcal{D}$ of allowable distributions in (*) to be the α-strongly regular distributions for a parameter $\alpha \in (0, 1]$.

1.2 Our Results

Our main result is that $m = \text{poly}(k, \frac{1}{\epsilon})$ samples are necessary and sufficient for the existence of an m-sample auction strategy that, for every strongly regular distribution $\mathbf{F}$, has expected revenue at least $(1 - \epsilon)$ times that of an optimal auction.

Both our upper and lower bounds on the sample complexity of revenue maximization are significant. For the lower bound, it is far from obvious that the number of samples per bidder needs to depend on k at all, let alone polynomially. Indeed, for many relaxations of the problem we study, the sample complexity is a function of ϵ only.

- If there is an unlimited supply of items (digital goods), then the problem reduces to separate single-bidder problems, for which $\text{poly}(\frac{1}{\epsilon})$ samples suffice for a $(1 - \epsilon)$-approximation for all regular distributions [8, Lemma 4.1].

- If bidders' valuations are independent and *identical* draws from an unknown regular distribution, then $\text{poly}(\frac{1}{\epsilon})$ samples suffice for a $(1 - \epsilon)$-approximation [8, Theorem 4.3].

- If only a $\frac{1}{2}$-approximation of the optimal expected revenue is required, then only a *single* sample is required. This follows from a generalization of the Bulow-Klemperer theorem [4] to non-i.i.d. bidders [13, Theorem 4.4].

$1 - \frac{1}{M}$. The optimal auction for such a distribution earns expected revenue M. It is not difficult to prove that, for every m, there is no m-sample auction strategy with near-optimal revenue for every such distribution — for sufficiently large M, all m samples are 0 w.h.p. and the auction strategy has to resort to an uneducated guess for M.

Thus, the necessary dependence on k fundamentally involves the interplay between bidder competition, non-identical distributions, and a very close (but still constant) approximation of the optimal revenue.

On a conceptual level, our lower bound shows that designing c-approximate auctions for constants c sufficiently close to 1 is a qualitatively different problem than for more modest constants like $\frac{1}{2}$. For example, previous work has demonstrated that auctions with reasonably good approximation factors are possible with minimal dependence on the valuation distributions (e.g. [5, 13]) or even, when there is no bidder with a unique valuation distribution, with *no* dependence on the valuation distributions [7, 8, 19]. Another interpretation of some previous results, such as [6, 5], is the existence of constant-factor approximate auctions that derive no benefit from bidder competition. Our lower bound identifies, for the first time, a constant approximation threshold beyond which "robustness" and "prior-independence" results of these types cannot extend. Our argument formalizes the idea that, with two or more non-identical bidders, the *only way* to achieve a sufficiently good constant approximation of the optimal revenue is through a detailed understanding of bidders' valuation distributions and an essentially optimal resolution of bidder competition.

We provide an upper bound on the number of samples needed for near-optimal approximation by analyzing a very natural auction. Recall that for a distribution $\mathbf{F}$ that is known a priori, Myerson's optimal auction gives the item to the bidder with the highest virtual valuation $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, or to no one if all virtual valuations are negative [16]. The *empirical Myerson auction* is the obvious analog when one has data rather than distributional knowledge: define $\bar{F}_i$ as the empirical distribution of the samples from F_i, and run the optimal auction for $\bar{\mathbf{F}}$.[3] We prove that a variant on the empirical Myerson auction has expected revenue at least $(1-\epsilon)$ times optimal provided it is given a sufficiently large polynomial number of samples.[4] A key aspect in our analysis is identifying the (non-pointwise) sense in which empirical virtual valuation functions approximate the actual virtual valuation functions; this is non-trivial even for the special case of MHR distributions.

1.3 Technical Approach

The proofs of our upper and lower bounds are fairly technical, so we provide here an overview of the main ideas. We begin with the upper bound, which roughly consists of the following steps.

1. (Lemma 6.3) For some fixed bidder with distribution F, consider the corresponding m samples $v_1 \geq \cdots \geq v_m$. Define the "empirical quantile" $\bar{q}_j$ of v_j as $\frac{2j-1}{2m}$, the expected quantile of the jth order statistic. Taking a "net" of quantiles and applying standard large deviation bounds shows that all but the top $\hat{\xi}$ fraction of the empirical quantiles are good multiplicative approximations of their expectations with high probability (w.h.p.); here $\hat{\xi} > 0$ is a key parameter that will depend on k and ϵ.

2. (Lemma 6.5) Recall that the expected revenue of an auction equals its expected virtual surplus [16]. Myerson's optimal auction maximizes virtual surplus pointwise, whereas our auction maximizes (ironed) empirical virtual surplus pointwise. In a perfect world, we would be able to argue that the empirical virtual valuation functions are good pointwise approximations of the true virtual valuation functions, and hence the expected virtual surplus of our auction is close to that of Myerson's auction. Unfortunately, good relative

[3] Since the empirical distributions are generally not regular even when the underlying distributions $\mathbf{F}$ are, a standard extra "ironing" step is required; see Section 2 for details.

[4] Left unmodified, the empirical Myerson mechanism can be led astray by poor approximations at the upper end of the valuation distributions caused by a small sample effect. We prove that excluding the very highest samples from the empirical distributions addresses the problem.

approximation of quantiles does not necessarily translate to good relative approximation of virtual valuations. The reason is that a virtual valuation function $v - \frac{1-F(v)}{f(v)}$ can change arbitrarily rapidly in a region where the density changes rapidly (even for MHR distributions).

We instead prove a different sense in which empirical virtual values approximate actual virtual values, working in the (quantile) domain as well as in the range of the virtual valuation functions. Recall that the quantile $q(v)$ is defined as $1 - F(v)$. We show that for suitable $\Delta_1, \Delta_2 > 0$, for all but the top $\hat{\xi}$ fraction of quantiles in $[0, 1]$, w.h.p. the empirical virtual value $\bar{\varphi}(q)$ is sandwiched between $\varphi(q(1 + \Delta_1))$ and $\varphi(q/(1 + \Delta_2))$, modulo small additive factors. (By $\varphi(q)$ we mean $\varphi(F^{-1}(1 - q))$.)[5] These additive factors are functions of $1/q$, as well as of k and ϵ, which complicates the analysis.

3. (Lemmas 6.10 and 6.11) Consider a fixed bidder i. By the previous step, up to an additive factor, we can lower bound the virtual surplus contributed by a bidder i with a quantile $q_i = 1 - F_i(v_i)$ outside the top $\hat{\xi}$ fraction in the empirical Myerson mechanism by the virtual value contributed by i in the optimal auction when it has a quantile of $q_i(1 + \Delta_1)$. Or not quite: an additional issue is that the empirical virtual valuation of a different bidder j with quantile q_j might be larger than its true virtual value, leading the empirical Myerson auction to allocate to j over the rightful winner i, and resulting in a reduction in the total virtual surplus being accummulated as compared to the actual Myerson auction. The difference in virtual values is bounded by the additive factor described in (2); as the outcomes are determined in the empirical auction, we will need two additive factors: for bidder i, the factor from the lower sandwiching bound, and for bidder j, the factor from the upper sandwiching bound. We will show that there is only a small probability of large additive factors, which suffices to bound the expected reduction in revenue when the additive factors are large. When the additive factors are small, their contribution to the reduction in revenue is also small. Both reductions end up being a polynomial function of k and ϵ.

4. (Lemma 6.8 and 6.9) There is one more issue. Because of the shift in quantile space — we compare the virtual value in the Myerson auction at quantile $q_i(1 + \Delta_1)$ to the virtual value in the empirical auction at quantile q_i — and also because the reserve prices in the two auctions may differ, we also have to analyze the revenue loss at the lower end of the distributions, or more precisely, around the reserve prices. This too is a polynomial function of k and ϵ.

We now proceed to the lower bound proof. This involves arguing that, if the number of samples is too small, then for every auction strategy, there exists a distribution for which the auction strategy's expected revenue is not near-optimal. We prove this by exhibiting a "distribution of distributions" and proving that every auction format has expected revenue — where the expectation is now with respect to both the initial random choice of the valuation distributions, and then with respect to both the m samples and the input — bounded away from the expected revenue of an optimal auction (where the expectation is over both the choice of distributions and the input). We are unaware of any other lower bounds in auction theory that have this form.

Our construction involves taking a base set of "worst-case" α-strongly regular distributions and truncating them at random points. A key observation is that, when such a distribution is truncated at a point H_i, the corresponding virtual valuation function is linear with coefficient

[5] The extent to which empirical virtual values approximate true virtual values has also been studied in other works, including [10].

α except at the truncation point, where the virtual valuation jumps to H_i. The high-level intuition is that, when confronted with valuations that are higher than those seen in any of the samples, no auction can know whether a high valuation v corresponds to a truncation point (with virtual value v) or not (with virtual value only $\alpha(v-1)$). Properly implemented, this idea can be used to prove that every auction strategy errs with constant probability on precisely the set of inputs that contribute the lion's share of the optimal revenue. The lower bound follows.

1.4 Additional Related Work

A few previous works study the convergence of an auction's revenue to the optimal revenue under different limits. These papers generally assume, unlike the present work, that bidders are symmetric and valuations are uniformly bounded from above. Neeman [17] considers single-item auctions with i.i.d. bidders, and quantifies the fraction of the optimal welfare extracted as revenue by the Vickrey auction, as a function of the number of bidders. Segal [20] and Baliga and Vohra [3] prove asymptotic optimality results for certain natural mechanisms when bidders are symmetric, goods are identical, and the number of bidders is large. Goldberg et al. [9] quantify the rate at which their RSOP auction approaches full optimality, with respect to a fixed-price benchmark, as a function of the number of winners under this benchmark.

It is intuitively clear that, by the law of large numbers, for every fixed distribution $\mathbf{F}$, the revenue of the empirical Myerson auction converges to that of the optimal auction provided the number of samples is sufficiently large. The only other paper we are aware of that proves, as we do, sample complexity bounds that apply uniformly to *all* distributions (subject to a tail condition) is [8]. [8] were motivated by "prior-independent" auctions that use the bids of a random subset of the (i.i.d.) bidders as samples to guide how to set prices for the rest of the bidders.[6] In analyzing these mechanisms, [8] solves the single-bidder version of the central problem studied in this paper. As our results show, the problem is quite different and more delicate with many non-i.i.d. bidders.

The problems that we study are clearly reminiscent of sample complexity questions that are common in learning theory (see e.g. [1]). A key difference is that we are interested only in optimizing a particular objective function (the expected revenue), and not in learning the underlying distribution per se. Differences aside, we expect rich connections between learning theory and auction theory to be developed in the near future (see also [2]).

2 The Empirical Myerson Auction

Preliminaries We begin by reviewing Myerson's optimal auction [16] for the case of known distributions. There are k bidders, and for each bidder B_i, $1 \leq i \leq k$, there is a distribution F_i from which its valuation is drawn.[7]

For each buyer B_i, the auctioneer computes a virtual valuation $\varphi_i(v) = v - [1 - F_i(v)]/f_i(v)$, where f_i is the density function corresponding to F_i. It is required that $\varphi_i(v)$ be a non-decreasing function of v (if this does not hold φ_i can be modified, *ironed*, so that it does hold, as implicitly explained in the next paragraph). Then the auctioneer essentially runs an analog of a second-price auction on the virtual values of the bids (virtual bids for short): the bidder, if any, with the highest non-negative virtual bid wins the auction (ties are broken arbitrarily) and is charged

[6]Similarly, our sample complexity upper bound naturally leads to a prior-independent single-item auction. This auction achieves a $(1 - \epsilon)$-approximation of the optimal auction when bidders' valuations are drawn from different regular distributions $F_1, \ldots, F_k$ and there are sufficiently many bidders of each type.

[7]Our results can be extended to the case of k groups of an arbitrary number of bidders, where all the bidders from group i have i.i.d. valuations drawn from F_i.

the minimum bid needed to win (or at least to tie for winning). More precisely, let B_i be the winning bidder and let b_2 be the second highest virtual bid. Then the price is ${\varphi_i}^{-1}(\min\{0, b_2\})$. We note that ${\varphi_i}^{-1}(0)$ can be viewed as a bidder specific reserve price for B_i; it is also called the *monopoly price* for B_i.

We can also describe the auction in terms of a revenue function. This also allows for situations where $\varphi_i(v)$ is not an increasing function of v. The revenue function is computed in quantile space: $q_i(v) = 1 - F_i(v)$ is the probability that B_i will have a valuation of at least v. Now we view v as a function of q_i. We introduce the expected revenue function, $R_i(q_i)$. It is a function of the quantile q_i: $R_i(q_i) = v(q_i) \cdot q_i$ is the expected revenue if B_i is the sole bidder and $v(q_i)$ is the price being charged. The auctioneer computes the smallest concave upper bound $\mathrm{CR}_i(q)$ of $R_i(q)$. Now $\varphi_i(v(q_i))$ is defined to be the slope of $\mathrm{CR}_i(q_i)$ (this yields an increasing φ_i, which is the same virtual value as before in the case that the earlier φ_i was non-decreasing). At points where there is no unique slope we choose $\varphi_i(q(v)) = \lim_{(r>q)\to q} \varphi_i(r)$. The auction then proceeds as before. Henceforth, overloading notation, we will write $\varphi_i(q_i)$ rather than $\varphi_i(q_i(v))$.

The Empirical Myerson Auction In the empirical Myerson auction, we assume we are given m independent samples from each distribution F_i. The empirical auction treats the resulting empirical distribution as the actual distribution in a Myerson auction; in our variant, a number of the samples with the highest values are discarded, and there is a further detail regarding how to handle any high bids that occur in the auction (i.e. bids larger than the largest non-discarded sample), discussed below.

Suppose that the m independent samples drawn from F_i have values $v_{i1} \geq v_{i2} \geq \ldots \geq v_{im}$. v_{ij} is treated as the value at empirical quantile $\frac{2j-1}{2m} = \bar{q}_i(\frac{2j-1}{2m})$. To construct the empirical revenue curve, $\overline{R}_i(\bar{q}_i)$, we define $\overline{R}_i(\frac{2j-1}{2m}) = \frac{2j-1}{2m} v_j$. Before constructing the convex hull $\overline{\mathrm{CR}}_i$ of $\overline{R}_i$ we first discard the $\lfloor \hat{\xi} m \rfloor - 1$ largest samples, for a suitable $\hat{\xi} > 0$, and add the point $\overline{R}_i(0) = 0$. Let $\bar{\xi} = \frac{2\hat{\xi}-1}{2m}$, the empirical quantile of the largest non-discarded sample. For each empirical quantile $\bar{q}_i \geq \bar{\xi}$, the corresponding revenue $\overline{\mathrm{CR}}_i(\bar{q})$ yields an *ironed* valuation, $\bar{v}_i(\bar{q}) = \overline{\mathrm{CR}}_i(\bar{q})/\bar{q}$. The slope of the convex hull at $\bar{q}$ is defined to be the empirical virtual value for ironed values $\bar{v}_i \leq v_{i,\hat{\xi}m}$. Note that $\bar{v}_i(\bar{\xi}) = v_{i,\hat{\xi}m}$. The reason for discarding the largest samples is that if they were present there is a non-negligible probability that they would create a poor approximation at the high value end of the distribution, which is the end that matters the most. Finally, for $v > v_{i,\hat{\xi}m}$, the virtual value is defined to be the (unironed) actual value.

We then run Myerson's auction on these constructed virtual values, that is, as before, the bidder, if any, with the highest non-negative virtual value wins (with ties broken arbitrarily), and pays an amount equal to the lowest bid needed to ensure a (tied) win.

Theorem 2.1 (Myerson)**.** *The expected revenue of any single item auction is given by* $\sum_i E[\varphi_i \cdot w_i]$ *where* $w_i(q_i)$ *is the probability that* B_i *wins the item with a bid at quantile* q_i *in* F_i.

Let $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ be a vector of quantiles drawn from $F_1 \times F_2 \times \ldots \times F_k$. We can rewrite the expected revenue bound as

$$\sum_i E[\varphi_i \cdot w_i] = \sum_i \int_{\mathbf{q}} \varphi_i(q_i) I_i(\mathbf{q})\, d\mathbf{q}, \tag{2}$$

where $I_i(\mathbf{q})$ is the indicator function showing whether B_i wins when the bids are at quantile $\mathbf{q}$ (or in the event of a tie, is the appropriate probability). This immediately implies that allocating to a bidder with the highest virtual value, i.e. Myerson's auction, is optimal.

3 The Results

We state upper and lower bound results in turn.

Theorem 3.1. *In the empirical Myerson auction with k bidders each having a regular distribution, using m independent samples from its distribution for each bidder, the resulting expected revenue satisfies* $\overline{\mathrm{MR}} \geq (1-\epsilon)\mathrm{MR}$ *if* $m = \Omega(\frac{k^{11}}{\epsilon^7}(\ln^3 k + \ln^3 \frac{1}{\epsilon}))$.

Our lower bound result has an analogous form although the polynomial in k and ϵ is smaller and does become larger for smaller α.

Theorem 3.2. *For every auction strategy Σ, for every $k \geq 2$, for every $\alpha > 0$, for sufficiently small $\epsilon > 0$, there exists a set $F_1, \ldots, F_k$ of α-strongly regular distributions such that the expected revenue of the auction strategy (over the samples and the input) is at most* $(1-\epsilon)\mathrm{MR}$, *if*

i. for $\alpha = 1$, $m \leq \left(\frac{1-\ln 2}{192e^3}\right)^{1/2} \frac{1}{\ln(k/\sqrt{\epsilon})} \frac{k}{\sqrt{\epsilon}}$,

ii. for $\alpha < 1$ *and* $\alpha^{1/(1-\alpha)} \geq \frac{1}{k}$, $m \leq \left(\frac{1-\alpha 2^{1-\alpha}}{192e^3}\right)^{1/2} \frac{k}{\epsilon^{1/(1+\alpha)}}$, *and*

iii. for $\alpha \leq \frac{1}{4}$ *and* $\alpha^{1/(1-\alpha)} = \frac{\delta}{k}$, $m \leq \frac{1}{384e^3} \frac{k}{\epsilon}$.

We prove these two theorems in Sections 6 and 5 respectively. Before that, in the next section, we briefly indicate another application of α-strong regularity.

4 Applications of Strong Regularity

We believe our definition of α-strongly regular distributions is of independent interest. Almost all previous Bayesian approximation guarantees are one of three types: for all distributions, for all regular distributions, or for all MHR distributions (see e.g. [11]). Strongly regular distributions interpolate between regular and MHR distributions, and should broaden the reach of many existing approximation bounds that are stated only for MHR distributions and are known to fail for general regular distributions. To prove this point, we mention a couple of examples of such extensions; we are confident that many others are possible.

Hartline et al. [14, Theorem 4.2] study a revenue maximization problem in social networks, and give a mechanism with approximation guarantee $e/(4e-2)$ when players' private parameters are distributed according to MHR distributions. The MHR assumption is used to argue that the probability of a sale at the monopoly price is at least $1/e$ [14, Lemma 4.1]. Lemma 4.1, given below, generalizes this to α-strongly regular distributions, and the approximation guarantee in [14] extends accordingly (with the term $1/e$ replaced by $1/\alpha^{1/(1-\alpha)}$).

Hartline and Roughgarden [13, Theorem 3.2] consider downward-closed single-parameter environments and prove that, when bidders' valuations are drawn from MHR distributions, the VCG mechanism with monopoly reserve prices has expected revenue at least $\frac{1}{2}$ times that of an optimal mechanism. With α-strongly regular distributions, the approximation guarantee degrades with decreasing α as $1/(1+\frac{1}{\alpha})$.

Lemma 4.1. *Let F be an α-strongly regular distribution with monopoly price r. Let $q(r)$ be the quantile of valuation r in distribution F.*

i. [14]. For $\alpha = 1$, $q(r) \geq \frac{1}{e}$.

ii. For $0 < \alpha < 1$, $q(r) \geq \alpha^{1/(1-\alpha)}$.

Proof. For (i) see Lemma 4.1 in [14]. We prove (ii). Let $h(v)$ denote the hazard rate. Recall that $\varphi(v) = v - 1/h(v)$. Define $H(x) = \int_0^x h(v)dv$. As in well known and easily verified, $q(v) = e^{-H(v)}$. α-strong-regularity, $\frac{d\phi}{dv} \geq \alpha$, implies that $1 + \frac{1}{h^2}\frac{dh}{dv} \geq \alpha$, or $-\frac{d}{dv}\left(\frac{1}{h}\right) \geq \alpha - 1$. For notational simplicity, we set $\lambda = 1 - \alpha$. Then, for $v \leq r$, $h(v) \leq \frac{1}{\lambda v + c}$ where $h(r) = \frac{1}{\lambda r + c}$. Now $\varphi(r) = 0$, so $h(r) = 1/r$. Thus $1/r = 1/(\lambda r + c)$, or $c = r(1-\lambda)$. This gives

$$h(v) \leq \frac{1}{\lambda(v-r)+r}.$$

We obtain

$$\begin{aligned} q(r) \;=\; e^{-H(r)} \;&=\; e^{-\int_0^r h(v)dv} \\ &\geq\; e^{-[\frac{1}{\lambda}\log(r+\lambda(v-r))\,|_0^r]} \\ &=\; e^{-\frac{1}{\lambda}\log\frac{r}{r(1-\lambda)}} \;=\; e^{\log(1-\lambda)^{1/\lambda}} \\ &=\; (1-\lambda)^{1/\lambda} \;=\; \alpha^{1/(1-\alpha)}. \end{aligned}$$

□

5 The Lower Bound

Formal Statement Fix $\alpha > 0$ and $0 < \delta \leq 1$, where δ is sufficiently small. The setting is a single-item environment. By an *auction strategy*, we mean a function Σ that takes as input $m = k/\delta$ valuation profiles, or "samples" (each sample is a k-vector) and outputs an auction $\mathcal{A}$. In our single-parameter setting, there is no loss of generality in assuming that $\mathcal{A}$ is a direct-revelation dominant-strategy incentive-compatible auction [16]. The revenue of an auction strategy on a sequence of $m+1$ valuation profiles $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m+1)}$ is defined as that of $\mathcal{A}$ on $\mathbf{v}^{(m+1)}$, where $\mathcal{A} = \Sigma(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m)})$ is the auction output by the strategy given the first m profiles.

We show that for every auction strategy Σ, there exists a set $F_1, \ldots, F_k$ of α-strongly regular distributions such that the expected revenue of the auction strategy (over the samples and the input) is at most the following fraction of the expected revenue of the optimal auction for $F_1, \ldots, F_k$:

$$1 - \epsilon(\alpha, \delta) = 1 - \frac{1 - \alpha 2^{1-\alpha}}{192e^3}\delta^{1/(1+\alpha)} \qquad \text{for } \alpha < 1 \text{ and } \alpha^{1/(1-\alpha)} \geq \tfrac{1}{k} \tag{3}$$

$$1 - \epsilon(\alpha, \delta) = 1 - \frac{1}{384e^3}\delta \qquad \text{for } \alpha \leq \frac{1}{4} \text{ and } \alpha^{1/(1-\alpha)} = \frac{\delta}{k} \tag{4}$$

$$1 - \epsilon(1, \delta) = 1 - \frac{1 - \ln 2}{192e^3 \ln\frac{k}{\delta}}\delta^2 \qquad \text{for } \alpha = 1 \tag{5}$$

We note that if $\alpha < 1$, then $\alpha 2^{1-\alpha} < 1$ also. Substituting k/m for δ yields the bounds in Theorem 3.2, namely for $\alpha < 1$ and sufficiently small constant $\epsilon > 0$, $\Omega(k/\epsilon^{1/(1+\alpha)})$ samples are necessary for a $(1-\epsilon)$-approximation, and for $\alpha \leq \frac{1}{4}$ and $\alpha^{1/(1-\alpha)} = \frac{1}{m}$, $\Omega(k/\epsilon)$ samples are necessary. For the MHR ($\alpha = 1$) case, $\Omega(k/\sqrt{\ln k/\epsilon}\sqrt{\epsilon})$ samples are necessary.

The Base Distributions We identify the worst-case distributions for a given $\alpha > 0$. Specifically, for $v \in [0, \infty)$, consider

$$F^\alpha(v) = 1 - \left(1 + \frac{1-\alpha}{\alpha} v\right)^{-\frac{1}{1-\alpha}} \qquad \text{for } 0 < \alpha < 1;$$

$$F^1(v) = 1 - e^{-v} \qquad \text{for } \alpha = 1;$$

$$f^\alpha(v) = \frac{1}{\alpha}\left(1 + \frac{1-\alpha}{\alpha} v\right)^{-\frac{2-\alpha}{1-\alpha}} \qquad \text{for } 0 < \alpha < 1;$$

$$f^1(v) = e^{-v}, \qquad \text{for } \alpha = 1.$$

The corresponding hazard rate is

$$h^\alpha(v) = \frac{1}{\alpha + (1-\alpha)v} \qquad \text{for } 0 < \alpha < 1;$$

$$h^1(v) = 1 \qquad \text{for } \alpha = 1;$$

with virtual valuation

$$\varphi^\alpha(v) = \alpha(v-1) \text{ for } 0 < \alpha \leq 1.$$

A quick calculation shows that

$$(\mathcal{F}^\alpha)^{-1}(q) = \begin{cases} \frac{\alpha}{1-\alpha}\left(\frac{1}{q}\right)^{1-\alpha} - 1 & \text{if } \alpha < 1 \\ \ln k & \text{if } \alpha = 1 \end{cases} \tag{6}$$

The Construction We define a distribution over distributions. Each bidder i is either type A or type B (50/50 and independently). The distribution of a type A bidder is F^α. For a type B bidder i, we draw q uniformly from the interval $[0, \frac{\delta}{k}]$ and set $H_i = (F^\alpha)^{-1}(1-q)$. We then define bidder i's distribution F_i as equal to F^α on $[0, H_i)$ with a point mass with the remaining probability $1 - F^\alpha(H_i)$ at H_i. These distributions are always α-strongly regular. An important point is that the virtual valuation of a type B bidder is given by

$$\varphi(v) = \begin{cases} \alpha(v-1) & \text{if } v < H_i \\ H_i & \text{if } v = H_i. \end{cases} \tag{7}$$

Let q_α denote the monopoly price in a 1-bidder auction. By Lemma 4.1, if $\alpha < 1$, $q_\alpha \geq 1/\alpha^{1/(1-/\alpha)}$, and $q_1 \geq 1/e$. Let $v^* = (F^\alpha)^{-1}\left(\max\{1-q_\alpha, \frac{k-1}{k}\}\right)$, the value corresponding to quantile $\min\{q_\alpha, \frac{1}{k}\}$ in F^α, and let $R^* = \min\{kq_\alpha, 1\} \cdot v^*$, k times the revenue at this quantile. From (6), we have that

$$v^* = \begin{cases} \frac{\alpha}{1-\alpha}\left[\max\left\{\frac{1}{q_\alpha}, k\right\}^{1-\alpha} - 1\right] & \text{if } \alpha < 1 \\ \ln \max\left\{\frac{1}{q_\alpha}, k\right\} & \text{if } \alpha = 1 \end{cases} \tag{8}$$

Lemma 5.1. (Upper Bound on Optimal Revenue). *The expected revenue (over* $\mathbf{v}$*) of the optimal auction (w.r.t. the* H_i*'s) is at most* R^*.

Proof. First, the expected revenue of the optimal auction is upper bounded by that of the optimal auction for the case where all H_i's are $+\infty$ — i.e., where $F_i = F^\alpha$ for every i. (This follows because F^α stochastically dominates F_i for any H_i, so an optimal auction for the latter does at least as well for the former.) Second, by symmetry, when bidders valuations are i.i.d.

draws from F^α, every bidder has the same sale probability q in the (symmetric) optimal auction, and since there is only one item, this sale probability q is at most $\frac{1}{k}$; it is also at most q_α. Third, we obtain an upper bound by dropping the constraint of selling only one item and instead optimally selling to each bidder with probability at most q. Fourth, this is precisely k times the revenue of selling to a single bidder with valuation from F^α using the posted price $(F^\alpha)^{-1}(1-q)$. Fifth, by regularity, selling to a single bidder with posted price $(F^\alpha)^{-1}(1-q)$ with $q \le \min\{q_\alpha, \frac{1}{k}\}$ is therefore no better than selling with the posted price $v^* = (F^\alpha)^{-1}(\max\{1-q_\alpha, \frac{k-1}{k}\})$. The expected revenue from any one bidder is therefore at most the sale probability times v^*, namely $\min\{q_\alpha, \frac{1}{k}\} \cdot v^*$. The overall revenue, with k bidders, is thus at most $k \times \min\{q_\alpha, \frac{1}{k}\} \cdot v^* = R^*$, as claimed. □

Overview The high-level plan is the following. Fix an arbitrary auction strategy. Think of the random choices as occuring in three stages: in the first stage, the F_i's are chosen; in the second stage, m sample valuation profiles $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m)}$ are chosen (iid from $F_1 \times \cdots \times F_k$); in the final stage, the input $\mathbf{v}$ is chosen (independently from $F_1 \times \cdots \times F_k$). We prove that the expected revenue of the auction strategy (w.r.t. all three stages of randomness) is at most $1 - \epsilon(\alpha, \delta)$ times that of the optimal auction (w.r.t. all three stages or, equivalently, the first and third stages only).[8] Again, $1 - \epsilon(\alpha, \delta) < 1$ will be independent of k. This implies that, for every auction strategy, there exists a choice of $F_1, \ldots, F_k$ such that the expected revenue of the auction strategy is at most $1 - \epsilon(\alpha, \delta)$ times the expected revenue of the optimal auction for the distributions $F_1, \ldots, F_k$.

By Lemma 5.1, R^* is an upper bound on the optimal auction's expected revenue (equivalently, expected virtual surplus) for every choice of $F_1, \ldots, F_k$. The main argument is the following: there is an event $\mathcal{E}$ such that, for every auction strategy:

(i) the probability of $\mathcal{E}$ (over all three stages of randomness) is lower bounded by a function $g(\delta)$ of δ (and independent of k and α);

(ii) given $\mathcal{E}$, the expected virtual surplus of the auction strategy is at least $h(\alpha, \delta)R^*$ smallest than that of the optimal auction, where $h(\alpha, \delta) > 0$ is a function of α and δ only.

Since by (2), for each set of bids, the virtual surplus earned by the optimal auction is always at least that of the auction strategy, (i)–(ii) imply that the expected virtual surplus (and hence revenue) of the optimal auction exceeds that of the auction strategy by cR^* for some $c > 0$ depending on α and δ. Since OPT is at most R^*, on setting $\epsilon(\alpha, \delta) = c$, this implies the auction strategy's expected revenue is at most $1 - \epsilon(\alpha, \delta)$ times optimal.

The Main Argument To define $\mathcal{E}$, we use the principle of deferred decisions. We can flip the second- and third-stage coins before those of the first stage by sampling quantiles — $(m+1)n$ iid draws $\{q_i^{(j)}\}$ from the uniform distribution on [0,1]. (Once the distributions are chosen in the first stage, the valuation $v_i^{(j)}$ is just $F_i^{-1}(1 - q_i^{(j)})$.) We further break the first-stage coin flips into two substages; in the first, we determine bidder types (A and B); in the second, we choose H_i's for the type-B bidders. The event $\mathcal{E}$ is defined as the set of coin flips (across all stages) that meet the following criteria:

(P1) There are exactly two quantiles of the form $q_i^{(m+1)}$ that are at most $\frac{\delta}{k}$, say of bidders j and ℓ;

[8] Actually, we prove this about expected virtual surplus rather than expected revenue, but this is equivalent [16].

(P2) $q_j^{(m+1)}$ and $q_\ell^{(m+1)}$ are at least $\frac{\delta}{2k}$.

(P3) for $i = 1, 2, \ldots, m$, $q_j^{(i)}$ and $q_\ell^{(i)}$ are greater than $\frac{\delta}{k}$.

(P4) one of the bidders j, ℓ is type A, the other is type B (we leave random which is which).

(P5) the type B bidder (from among j, ℓ) has valuation equal to the maximum valuation from its distribution.

Lemma 5.2 (Statement (i))**.** *the probability of $\mathcal{E}$ (over all three stages of randomness) is lower bounded by a function $\gamma(\delta)$ of δ (and independent of k and α), where $\gamma(\delta) = \frac{\delta^2}{64e^3}$.*

Proof. We first sample the k quantiles corresponding to the second stage. Elementary computations show that property (P1) holds with probability at least $\frac{1}{2e}\delta^2$ (independent of α and k). Conditioned on (P1) holding, (P2) holds with probability $\frac{1}{4}$. (P3) is independent of the first two properties and holds with constant probability of at least $\frac{1}{e^2}$ (independent of α, k). (P4) is independent of the first three properties and holds with 50% probability. Conditioned on (P1), (P2), and (P4) (as (P3) is irrelevant), the probability of (P5) equals the probability that a uniform draw from $[0, \frac{\delta}{k}]$ (used to determine the H-value) is at least the q-value of the type B bidder, which is conditionally distributed uniformly on $[\frac{\delta}{2k}, \frac{\delta}{k}]$. This happens with probability $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$. We conclude that all of (P1)–(P5) hold with a positive probability, namely

$$\gamma(\delta) = \frac{\delta^2}{64e^3}.$$

□

We've reduced statement (ii) to the statement that, for every auction strategy, conditioned on $\mathcal{E}$, the strategy fails to allocate the item to the optimal bidder — the type-B bidder with its maximum-possible valuation — with constant probability. It suffices to analyze the auction strategy that, conditioned on $\mathcal{E}$, maximizes the probability (over the remaining randomness) of allocating to the optimal bidder — of guessing, from among the two bidders j, ℓ that in $\mathbf{v}^{(m+1)}$ have valuation at least $(F^\alpha)^{-1}(1-\frac{\delta}{k})$, which one is type A and which one is type B. Since the two bidders were symmetric ex ante, Bayes' rule implies that the probability of guessing correctly (given $\mathcal{E}$) is maximized by, for every $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m+1)}$, choosing the scenario that maximizes the likelihood of the valuation profiles $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m+1)}$ (given $\mathcal{E}$).

Lemma 5.3. *Every auction strategy, conditioned on $\mathcal{E}$, allocates to a non-optimal bidder with probability at least $\frac{1}{3}$.*

Proof. The only valuations that affect the relative likelihoods of the two scenarios are $v_j^{(m+1)}$ and $v_\ell^{(m+1)}$. We already know the optimal bidder is either j or ℓ. Property (P3) of event $\mathcal{E}$ implies that the m sample valuations from j and ℓ are equally likely to be generated under the two scenarios — the distributions of type-A and type-B bidders differ only for quantiles in $[0, \frac{\delta}{k}]$.

Now, given $v_j^{(m+1)}$ and $v_\ell^{(m+1)}$, the posterior probabilities of the two scenarios are *not* equal. The reason is that, conditioned on $\mathcal{E}$, the type-A bidder's valuation is distributed according $(F^\alpha)^{-1}(q)$ where q is uniform in $[\frac{\delta}{2k}, \frac{\delta}{k}]$, while the type-B bidder's valuation is distributed according to the smaller of two iid such samples.[9] Thus, assigning the item to the bidder of

[9] In more detail, consider a type-B bidder i and condition on the event that its quantile $q_i = 1 - F_i(v_i)$ is in $[\frac{\delta}{2k}, \frac{\delta}{k}]$ and that its valuation is its maximum possible, which is equivalent to the condition that its fictitious quantile q_i' that generates its threshold H_i lies in $[q_i, \frac{\delta}{k}]$. The joint distribution of (q_i, q_i') is the same as the process that generates two iid draws from $[\frac{\delta}{2k}, \frac{\delta}{k}]$ and assigns q_i and q_i' to the smaller and larger one, respectively. Note that the valuation of the bidder is, by definition, $(F^\alpha)^{-1}(1 - q_i')$.

j, ℓ with the lower valuation (in $\mathbf{v}^{(m+1)}$) maximizes the probability of allocating to the optimal (type-B) bidder. The probability that this allocation rule erroneously allocates the item to the type-A bidder is the probability that a sample for a distribution (the type-A bidder) is smaller than the minimum of two other samples from the same distribution (the type-B bidder), which is precisely $\frac{1}{3}$. □

Lemma 5.4. *The revenue of any sample-based auction strategy is at most the following fraction of an optimal auction's revenue:*

$$\begin{array}{ll} 1 - \dfrac{1}{192e^3}\dfrac{1}{\ln k/\delta}\delta^2 & \textit{if } \alpha = 1 \\ 1 - (1-\alpha 2^{1-\alpha})\dfrac{1}{192e^3}\delta^{1+\alpha} & \textit{if } \alpha < 1 \textit{ and } q_\alpha = \alpha^{1/(1-\alpha)} \geq \dfrac{1}{k} \\ 1 - \dfrac{1}{384e^3}\delta & \textit{if } \alpha \leq \dfrac{1}{4} \textit{ and } q_\alpha = \alpha^{1/(1-\alpha)} = \dfrac{\delta}{k}. \end{array}$$

Proof. Conditioned on $\mathcal{E}$, the virtual value for the type B item is at least

$$\phi_B \;\geq\; \begin{cases} \frac{\alpha}{1-\alpha}\left[\left(\frac{k}{\delta}\right)^{1-\alpha} - 1\right] & \text{if } \alpha < 1 \\ \ln\frac{k}{\delta} & \text{if } \alpha = 1 \end{cases}$$

and for the type A item is at most

$$\phi_A \;\leq\; \begin{cases} \alpha\left[\frac{\alpha}{1-\alpha}\left[\left(\frac{2k}{\delta}\right)^{1-\alpha} - 1\right] - 1\right] \;=\; \frac{\alpha(\alpha\cdot 2^{1-\alpha})}{1-\alpha}\left(\frac{k}{\delta}\right)^{1-\alpha} - \frac{\alpha}{1-\alpha} & \text{if } \alpha < 1 \\ \ln\frac{2k}{\delta} - 1 & \text{if } \alpha = 1 \end{cases}$$

Thus, conditioned on $\mathcal{E}$,

$$\varphi_B - \varphi_A \;\geq\; \begin{cases} \frac{\alpha}{1-\alpha}\left(\frac{k}{\delta}\right)^{1-\alpha}\left(1 - \alpha\cdot 2^{1-\alpha}\right) & \text{if } \alpha < 1 \\ 1 - \ln 2 & \text{if } \alpha = 1 \end{cases}$$

We now bound the fractional loss of revenue. By Lemma 5.2, $\mathcal{E}$ occurs with probability at least $\delta^2/(64e^3)$. By Lemma 5.3, conditioned on $\mathcal{E}$, a type A rather than a type B bidder is wrongly allocated the item with probability $\frac{1}{3}$. Thus the expected loss of revenue is at least

$$\frac{1}{3}\frac{\delta^2}{64e^3}(\varphi_B - \varphi_A).$$

By Lemma 5.1, the optimal revenue is at most R^*. Consequently, we can lower bound the fractional loss of revenue at follows.

If $\alpha < 1$ and $q_\alpha \geq \frac{1}{k}$, then the fractional loss of revenue is at least

$$\frac{\delta^2}{3\cdot 64e^3}\frac{\varphi_B - \varphi_A}{R^*} \;=\; \frac{1}{3\cdot 64e^3}(1-\alpha 2^{1-\alpha})\frac{\frac{\alpha}{1-\alpha}\left[\left(\frac{k}{\delta}\right)^{1-\alpha}\right]\delta^2}{\frac{\alpha}{1-\alpha}(k^{1-\alpha}-1)} \;\geq\; \frac{1-\alpha 2^{1-\alpha}}{192e^3}\delta^{1+\alpha}.$$

If $\alpha < 1$ and $\frac{\delta}{k} \leq q_\alpha \leq \frac{1}{k}$, then the fractional loss of revenue is at least

$$\frac{1}{3\cdot 64e^3}(1-\alpha 2^{1-\alpha})\frac{\frac{\alpha}{1-\alpha}\left[\left(\frac{k}{\delta}\right)^{1-\alpha}\right]\delta^2}{\frac{\alpha}{1-\alpha}kq_\alpha\left[\left(\frac{1}{q_\alpha}\right)^{1-\alpha} - 1\right]} \;\geq\; \frac{(1-\alpha 2^{1-\alpha})\left(\frac{k}{\delta}\right)^{1-\alpha}\delta^2}{192e^3 k(q_\alpha)^\alpha}$$

$$= \frac{(1-\alpha 2^{1-\alpha})\delta^{1+\alpha}}{192e^3(q_\alpha)^\alpha k^\alpha}.$$

In particular, if $q_\alpha = \frac{\delta}{k}$, then the fractional loss of revenue is at most $\frac{(1-\alpha 2^{1-\alpha})}{192e^3}\delta$. If in addition $\alpha \le \frac{1}{4}$, then $(1-\alpha 2^{1-\alpha}) \ge \frac{1}{2}$. Thus the fractional loss of revenue is at most $\frac{1}{384e^3}\delta$.

Finally, if $\alpha = 1$, as $q_\alpha \ge \frac{1}{e}$, the fractional loss of revenue is at most

$$\frac{1-\ln 2}{192e^3 \ln \frac{k}{\delta}}\delta^2.$$

□

Finally, we note that Lemma 5.4 proves (3), (4) and (5).

6 The Upper Bound

We begin by specifying notation so as to clearly distinguish parameters for Myerson's optimal auction from those for the empirical auction, as our analysis will be repeatedly comparing these two auctions. After a couple of simple results, Lemma 6.3 bounds the empirical quantiles as a function of the actual quantiles, and vice-versa (this is essentially Lemma 4.1 in [8]). Next, Lemma 6.5 relates the empirical and actual virtual values. With these in hand, in Section 6.5, we bound the expected revenue loss due to using the empirical auction as opposed to Myerson's optimal auction, assuming for the latter auction that the actual distributions were fully known.

6.1 Notation

Myerson's Auction Let MR (the "Myerson Revenue") denote the expected revenue recovered by Myerson's auction. Let $x_i(q)$ denote the probability that bidder B_i wins in Myerson's auction with a bid that has quantile q in its value distribution. Recall that $v_i(q)$ denotes the value corresponding to quantile q and $\varphi_i(q)$ denotes the virtual value at quantile q. Let $\text{MR}_i = E[\varphi_i \cdot x_i]$ denote the expected revenue provided by B_i in Myerson's auction. Let $q_i(v)$ denote the minimum quantile for value v; sometimes it will be convenient to let q_v^i denote $q_i(v)$, and to reduce clutter, we suppress the index i when it is clear from the context. Let r_i be the reserve price applied to B_i in Myerson's auction, namely the largest value for which $\varphi_i(q_i(v)) = 0$. Let q_{r_i} denote $q_i(r_i)$. Let $\text{SR}_i = E[\varphi_i(q) \,|\, q \ge r_i] = q_{r_i} \cdot r_i$; note that SR_i is the expected revenue if B_i were the only participant in Myerson's auction (SR_i is short for "Single buyer Revenue"). Sometimes, to reduce clutter, we suppress the index i and write SR instead of SR_i.

The Empirical Auction The empirical auction is defined in terms of the empirical quantile $\bar{q}$, but its analysis will entail considering its revenue as a function of the actual quantile q. We specify notation which will distinguish between these two parameters.

Let $\overline{\text{MR}}$ denote the expected revenue recovered by the empirical auction. For empirical quantiles $\bar{q} \ge \bar{\xi}$, given a price $\bar{v}_i(\bar{q})$, the ironed value at quantile $\bar{q}$, let $\overline{\text{CR}}_i(\bar{q})$ denote the revenue expected from B_i, assuming B_i were the only participant in the empirical auction; $\overline{\text{CR}}_i(\bar{q}) = \bar{v}_i(\bar{q}) \cdot \bar{q}$. Let $\bar{\varphi}_i(\bar{q})$ denote the corresponding empirical virtual value as a function of $\bar{q}$. Let $\bar{x}_i(\bar{q})$ denote the probability that bidder B_i wins in the empirical auction with the bid $\bar{v}_i(\bar{q})$. $\bar{q}_i(v) = \bar{v}_i^{-1}(v)$ denotes the predicted quantile at value v (in the event this is not uniquely defined we choose it to be the minimum such quantile). We also write $\bar{q}_{iv}$ for this quantile, and sometimes suppress the index i when it is clear from the context. $\bar{r}_i$ will denote the empirical reserve price, which is the minimum of $\bar{v}_i(\bar{\xi})$ and the largest value $\bar{v}$ for which $\bar{\varphi}_i(\bar{q}_i(\bar{v})) = 0$, and $\bar{q}_{\bar{r}_i}$ will denote the corresponding empirical quantile. Sometimes, to reduce clutter, we suppress the index i and write $\overline{\text{SR}}(q)$, $\bar{\varphi}(q)$, and $\bar{q}$ for these functions.

The actual quantile q corresponding to empirical quantile $\bar{q}$ is defined by the relation $v_i(q) = \bar{v}_i(\bar{q})$; it is denoted by $q(\bar{v}_i(\bar{q}))$; we write it as q for short. Finally, we write the empirical probability of winning as $\tilde{x}_i(q) = \bar{x}_i(\bar{q})$.

6.2 Two Simple Results

Claim 6.1. *i.* $\mathrm{MR} = \sum_{i=1}^{n} \mathrm{MR}_i$.

ii. $\mathrm{SR}_i \le \mathrm{MR}$ *for all* i.

Lemma 6.2. *Let F be a regular distribution. Let $q(r) \ge q_1 > q_2$, where $q(r)$ is the quantile of the reserve price for F. Then $v(q_2) \le \frac{q_1}{q_2} v(q_1)$.*

Proof. $\varphi(q) \ge 0$ for $q \le q(r)$, and consequently $R(q_1) \ge R(q_2)$. $R(q_1) = q_1 \cdot v(q_1)$ and $R(q_2) = q_2 \cdot v(q_2)$. Thus $q_1 \cdot v(q_1) \ge q_2 \cdot v(q_2)$, and the result follows. □

6.3 Relating the Actual and Empirical Quantiles

The following result is essentially Lemma 4.1 in [8].

Lemma 6.3. *Let F be a regular distribution. Suppose m independent samples with values $v_1 \ge v_2 \ge \ldots \ge v_m$ are drawn from F. Let $\gamma > 0$, $\hat{\xi} = \frac{k}{m} < 1$ for some integer $k > 0$ be given, and let ν be defined by $1 + \nu = (1+\gamma)^2$. Let $t_j = \frac{2j-1}{2m}$. Then, for all $v \le v_{\hat{\xi} m}$,*

$$q(v) \in \left[\frac{\bar{q}(v)}{(1+\gamma)^2}, \bar{q}(v)(1+\gamma)^2\right] = \left[\frac{\bar{q}(v)}{(1+\nu)}, \bar{q}(v)(1+\nu)\right]$$

or equivalently

$$\bar{q}(v) \in \left[\frac{q(v)}{(1+\nu)}, q(v)(1+\nu)\right]$$

with probability at least $1-\delta$, if $\gamma\hat{\xi}m \ge 1$, $(1+\gamma)^2 \le \frac{3}{2}$, and $m \ge \frac{6}{\gamma^2(1+\gamma)\hat{\xi}} \max\{\frac{\ln 3}{\gamma}, \ln\frac{6}{\delta}\}$.

Proof. We begin by identifying a subsequence of the samples, $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$, with $j_1 \le j_2 \le \ldots \le j_k$; we rename the sequence $u_1, u_2, \ldots, u_k$ for notational ease. It will be the case that $u_{l+1} \le (1+\gamma)u_l$ and $t_{j_k}(1+\gamma) > 1$. We will show that

$$q(v_j) \in \left[\frac{t_j}{(1+\gamma)}, t_j(1+\gamma)\right], \qquad \text{for } v_j \in U = \{u_1, \ldots, u_k\}.$$

The claimed bound is then immediate as either each $v \le v_{\hat{\xi}m}$ is sandwiched between two items in U, or it is at most u_k.

We define the j_i, as follows: $j_1 = \hat{\xi}m$ and $j_{i+1} = \lfloor (1+\gamma)j_i \rfloor$ if $\lfloor (1+\gamma)j_i \rfloor \le m$, and otherwise j_{i+1} is not defined (i.e. $i = k$). As $\gamma\hat{\xi}m \ge 1$, and $\gamma < 1$, $\lfloor (1+\gamma)\hat{\xi}m \rfloor \ge \hat{\xi}m + \lfloor \gamma\hat{\xi}m \rfloor \ge j_1 + 1$, from which we conclude that the sequence is strictly increasing and hence well defined.

Next we bound the probability that $q(u_i) > (1+\gamma)t(u_i)$, where $t(u_i)$ is defined by $u_i = v_{j_i}$ and $t(u_i) = t_{j_i}$. Now $q(u_i) > (1+\gamma)t(u_i)$ only if fewer than j_i samples have q values that are at most $(1+\gamma)t(u_i)$. As the expected number of such samples is $(1+\gamma)t(u_i)m$, a Chernoff bound gives the following upper bound on the probability that $q(u_i) > (1+\gamma)t(u_i)$ (cf. [15]):

$$\exp\{-\frac{\gamma^2(1+\gamma)t(u_i)m}{3}\}.$$

Similarly, the probability that $q(u_i) < t(u_i)/(1+\gamma)$ is bounded by

$$\exp\{-\frac{\gamma^2 t(u_i)m}{2(1+\gamma)}\}.$$

It will be helpful to bound both $t(u_1)m$ and $[t(u_{i+1}) - t(u_i)]m$. As $\hat{\xi}m \geq 1$, $t(u_1)m = (2\hat{\xi}m - 1)/2 \geq \frac{1}{2}\hat{\xi}m$. And as $\gamma\hat{\xi}m \geq 1$, $[t(u_{i+1}) - t(u_i)]m \geq \lfloor (1+\gamma)j_i \rfloor - j_i \geq \lfloor \gamma j_i \rfloor \geq \lfloor \gamma\hat{\xi}m \rfloor \geq \frac{1}{2}\gamma\hat{\xi}m$.

Now, by the union bound applied to all the $q(u_i)$, we obtain a failure probability of at most:

$$\begin{aligned}
&\sum_{i=1}^{k} \exp\{-\frac{\gamma^2(1+\gamma)t(u_i)m}{3}\} + \exp\{-\frac{\gamma^2 t(u_i)m}{2(1+\gamma)}\} \\
&\leq\ 2\sum_{i=1}^{k} \exp\{-\frac{\gamma^2(1+\gamma)t(u_i)m}{3}\} \quad \text{as } (1+\gamma)^2 \leq \frac{3}{2} \\
&\leq\ 4\sum_{i=0}^{k-1} \exp\left\{-\frac{\gamma^2(1+\gamma)[\hat{\xi}m + (i-1)\gamma\hat{\xi}m]}{6}\right\} \\
&\qquad \text{using the bounds on } t(u_1)m \text{ and } [t(u_{i+1}) - t(u_i)]m \\
&\leq\ \frac{4\exp\{-\frac{\gamma^2(1+\gamma)\hat{\xi}m}{6}\}}{1-\exp\{-\frac{\gamma^3(1+\gamma)\hat{\xi}m}{6}\}} \ \leq\ 6\exp\left\{-\frac{\gamma^2(1+\gamma)\hat{\xi}m}{6}\right\} \\
&\qquad \text{if } \exp\left\{-\frac{\gamma^3(1+\gamma)\hat{\xi}m}{6}\right\} \leq \frac{1}{3}.
\end{aligned}$$

We want the failure probability to be at most δ. So we need $\frac{\gamma^2(1+\gamma)\hat{\xi}m}{6} \geq \ln\frac{6}{\delta}$, i.e. $m \geq \frac{6}{\gamma^2(1+\gamma)\hat{\xi}}\ln\frac{6}{\delta}$. We also need $m \geq \frac{6}{\gamma^3(1+\gamma)\hat{\xi}}\ln 3$ to satisfy the condition in the final inequality. $\square$

6.4 Relating the Actual and the Empirical Virtual Values

Let $\mathcal{E}_a$ be the event that the high probability outcome of Lemma 6.3 occurs, namely that for all $v \leq v_{\lfloor \hat{\xi}m \rfloor}$, $q(v) \in \left[\frac{\bar{q}(v)}{(1+\nu)}, \bar{q}(v)(1+\nu)\right]$. $\mathcal{E}_a$ occurs with probability at least $1-\delta$. It will also be helpful to express the bound on v as a bound on $\bar{q}$. To this end, we define $\bar{\xi} = t_1 = \frac{2\hat{\xi}-1}{2m}$.

We will repeatedly encounter terms of the form $\varphi(\lambda q)$ with $\lambda > 1$; For $\lambda q > 1$, $\varphi(\lambda q)$ is interpreted to mean $\varphi(1)$; similarly for $\bar{\varphi}$.

Lemma 6.4. *Conditioned on $\mathcal{E}_a$, for all empirical quantiles $\bar{q} \geq \bar{\xi}$, $\overline{CR}(\bar{q}) \leq \bar{q} \cdot v(\frac{\bar{q}}{1+\nu})$, and for all $t_j = \frac{2j-1}{2m} \geq \bar{\xi}$, $\overline{CR}(t_j) \geq t_j \cdot v(\ t_j(1+\nu))$.*

This lemma is not as obvious as it may seem for it concerns points on the convex hull $\overline{\text{CR}}$ of the set of points $\bar{R}$ that are used to specify the empirical revenue.

Proof. By Lemma 6.3, as $\mathcal{E}_a$ holds, for all $t_j \geq \bar{\xi}$,

$$t_j \cdot v(t_j(1+\nu)) \leq \bar{R}(t_j) \leq t_j \cdot v(\frac{t_j}{1+\nu}).$$

We define $\bar{L}(\bar{q}) = \bar{q} \cdot v(\bar{q}(1+\nu))$ and $\bar{U}(\bar{q}) = \bar{q} \cdot v(\frac{\bar{q}}{1+\nu})$ for all $\bar{q}$.

Note that for any pair $q \neq q'$ of quantiles, the line joining the actual revenue $R(\frac{q}{1+\nu}) = \frac{q}{1+\nu}v(\frac{q}{1+\nu})$ to $R(\frac{q'}{1+\nu}) = \frac{q'}{1+\nu}v(\frac{q'}{1+\nu})$ is parallel to the line joining $\bar{U}(q)$ to $\bar{U}(q')$, for the latter line is obtained by expanding the former line by a factor $1+\nu$ in both the quantile and revenue dimensions. By the regularity of φ, the curve defined by R is convex, and consequently, the points $\bar{U}(\bar{q})$ all lie on their convex hull.

For $t_j \geq \bar{\xi}$, $\bar{U}(t_j)$ is an upper bound on $\bar{R}(t_j)$; it follows that the convex hull for the empirical revenue, for $\bar{q} \geq \bar{\xi}$, is enclosed by the convex hull $\bar{U}(\bar{q})$, and consequently $\overline{\mathrm{CR}}(\bar{q}) \leq \bar{U}(\bar{q}) = \bar{q} \cdot v(\frac{\bar{q}}{1+\nu})$.

For the second result, the lower bound, we use a similar argument, but it will apply just to the empirical quantiles $t_j \geq \bar{\xi}$. Now, for any pair $q \neq q'$ of quantiles, the line joining the actual revenue $R(\frac{q}{1+\nu}) = \frac{q}{1+\nu} v(\frac{q}{1+\nu})$ to $R(\frac{q'}{1+\nu}) = \frac{q'}{1+\nu} \cdot v(\frac{q'}{1+\nu})$ is parallel to the line joining $\bar{L}(q)$ to $\bar{L}(q')$, and hence the points $\bar{L}(\bar{q})$ all lie on their convex hull. But, for $t_j \geq \bar{\xi}$, $\bar{L}(t_j) \leq \bar{R}(t_j)$, and consequently the values $\bar{R}(t_j)$ all lie on or above the curve $\bar{L}(t_j)$. □

The following lemma, which lies at the heart of out analysis, shows that w.h.p. $\varphi(q)$ is close to some value $\bar{\varphi}(\bar{q}')$ with $\bar{q}' \in [\frac{\bar{q}}{(1+\Delta)(1+\nu)^3}, \bar{q}(1+\Delta)(1+\nu)]$.

Lemma 6.5. *Let F be a regular distribution. Suppose that $(1+\Delta) \geq (1+\nu)^2$. Let $t_j = \frac{2j-1}{2m}$, for $1 \leq j \leq m$. Conditioned on $\mathcal{E}_a$, if $t_{j-1} < \bar{q} \leq t_j$, then*

i. for all $\bar{q} \geq \bar{\xi}(1+\Delta)(1+\nu)^3$, $\varphi(q) \leq \varphi(\frac{t_j}{(1+\nu)^2}) \leq \bar{\varphi}(\frac{\bar{q}}{(1+\Delta)(1+\nu)^3}) + 2(1+\Delta)(1+\nu)^3 \frac{\mathrm{SR}}{\bar{q}} \frac{\nu}{\Delta}$, and

ii. for all $\bar{q} \geq \bar{\xi}$, $\bar{\varphi}(\bar{q}(1+\Delta)(1+\nu)) \leq \varphi(t_j(1+\nu)) + 2(1+\nu)^2 \frac{\mathrm{SR}}{\bar{q}} \frac{\nu}{\Delta} \leq \varphi(q) + 2(1+\nu)^2 \frac{\mathrm{SR}}{\bar{q}} \frac{\nu}{\Delta}$.

Proof. The main part of the proof concerns the second inequality in (i) and the first one in (ii). We begin by proving the inequality in (i). First we give an upper bound on $\varphi(\frac{t_j}{(1+\nu)^2})$ and a lower bound on $\bar{\varphi}(\frac{\bar{q}}{(1+\Delta)(1+\nu)^3})$.

As F is regular, R is convex; thus:

$$
\begin{aligned}
\varphi(\frac{t_j}{(1+\nu)^2}) &\leq \frac{R(\frac{t_j}{(1+\nu)^2}) - R(\frac{t_j}{(1+\Delta)(1+\nu)^4})}{\frac{t_j}{(1+\nu)^2} - \frac{t_j}{(1+\Delta)(1+\nu)^4}} \\
&= \frac{\frac{t_j}{(1+\nu)^2} \cdot v(\frac{t_j}{(1+\nu)^2}) - \frac{t_j}{(1+\Delta)(1+\nu)^4} v(\frac{t_j}{(1+\Delta)(1+\nu)^4})}{\frac{t_j}{(1+\nu)^2} - \frac{t_j}{(1+\Delta)(1+\nu)^4}} \\
&= \frac{(1+\Delta)(1+\nu)^2 v(\frac{t_j}{(1+\nu)^2}) - v(\frac{t_j}{(1+\Delta)(1+\nu)^4})}{2\nu + \nu^2 + \Delta(1+\nu)^2}.
\end{aligned}
$$

The following bound applies only when $\frac{t_j}{(1+\Delta)(1+\nu)^3} \geq \bar{\xi}$ for otherwise $\overline{\mathrm{CR}}(\frac{t_j}{(1+\Delta)(1+\nu)^3})$ is not defined; the constraint $\bar{q} \geq \bar{\xi}(1+\Delta)(1+\nu)^3$ suffices.

$$
\begin{aligned}
\bar{\varphi}(\frac{\bar{q}}{(1+\Delta)(1+\nu)^3}) &\geq \frac{\overline{\mathrm{CR}}(\frac{t_j}{(1+\nu)^3}) - \overline{\mathrm{CR}}(\frac{t_j}{(1+\Delta)(1+\nu)^3})}{\frac{t_j}{(1+\nu)^3} - \frac{t_j}{(1+\Delta)(1+\nu)^3}} \\
&\geq \frac{\frac{t_j}{(1+\nu)^3} \cdot v(\frac{t_j}{(1+\nu)^2}) - \frac{t_j}{(1+\Delta)(1+\nu)^3} v(\frac{t_j}{(1+\Delta)(1+\nu)^4})}{\frac{t_j}{(1+\nu)^3} - \frac{t_j}{(1+\Delta)(1+\nu)^3}} \quad \text{(by Lemma 6.4)} \\
&= \frac{(1+\Delta) v(\frac{t_j}{(1+\nu)^2}) - v(\frac{t_j}{(1+\Delta)(1+\nu)^4})}{\Delta}.
\end{aligned}
$$

Now, we combine the bounds:

$$
\begin{aligned}
&\varphi(\frac{t_j}{(1+\nu)^2}) - \bar{\varphi}(\frac{\bar{q}}{(1+\Delta)(1+\nu)^3}) \\
\leq\ & \frac{(1+\Delta)(-2\nu-\nu^2)v(\frac{t_j}{(1+\nu)^2}) + [(2\nu+\nu^2)+\Delta(1+\nu)^2-\Delta]v(\frac{t_j}{(1+\Delta)(1+\nu)^4})}{\Delta[2\nu+\nu^2+\Delta(1+\nu)^2]} \\
\leq\ & \frac{(1+\Delta)(2\nu+\nu^2)[v(\frac{t_j}{(1+\Delta)(1+\nu)^4}) - v(\frac{t_j}{(1+\nu)^2})]}{\Delta[2\nu+\nu^2+\Delta(1+\nu)^2]} \\
\leq\ & \frac{(2\nu+\nu^2)(1+\Delta)v(\frac{t_j}{(1+\nu)^2})[(1+\Delta)(1+\nu)^2-1]}{\Delta[2\nu+\nu^2+\Delta(1+\nu)^2]} \qquad \text{(by Lemma 6.2)} \\
\leq\ & \frac{\nu(2+\nu)(1+\Delta)v(\frac{t_j}{(1+\nu)^2})}{\Delta} \ \leq\ \frac{\nu(2+\nu)(1+\Delta)\mathrm{SR}}{\Delta\frac{t_j}{(1+\nu)^2}} \ \leq\ 2(1+\Delta)(1+\nu)^3\frac{\nu}{\Delta}\frac{\mathrm{SR}}{\bar{q}}.
\end{aligned}
$$

The second inequality in (ii) is shown similarly. We start with an upper bound on $\bar{\varphi}(\bar{q}(1+\Delta)(1+\nu))$ and a lower bound on $\varphi(t_j(1+\nu))$. The first bound applies only when $t_j \geq \bar{\xi}$; here $\bar{q} \geq \bar{\xi}$ suffices.

$$
\begin{aligned}
\bar{\varphi}(\bar{q}(1+\Delta)(1+\nu)) \ \leq\ \bar{\varphi}(t_j(1+\Delta)) \ &\leq\ \frac{\overline{\mathrm{CR}}(t_j(1+\Delta)) - \overline{\mathrm{CR}}(t_j)}{(1+\Delta)t_j - t_j} \\
&\leq\ \frac{t_j(1+\Delta)\cdot v(\frac{t_j(1+\Delta)}{1+\nu}) - t_j v(t_j(1+\nu))}{\Delta t_j} \qquad \text{(by Lemma 6.4)} \\
&=\ \frac{(1+\Delta)v(\frac{t_j(1+\Delta)}{1+\nu}) - v(t_j(1+\nu))}{\Delta}.
\end{aligned}
$$

$$
\begin{aligned}
\varphi(t_j(1+\nu)) \ \geq\ \frac{R(\frac{t_j(1+\Delta)}{(1+\nu)}) - R(t_j(1+\nu))}{\frac{t_j(1+\Delta)}{(1+\nu)} - t_j(1+\nu)} \ &=\ \frac{\frac{t_j(1+\Delta)}{1+\nu}v(\frac{t_j(1+\Delta)}{1+\nu}) - t_j(1+\nu)v(t_j(1+\nu))}{\frac{t_j(1+\Delta)}{1+\nu} - t_j(1+\nu)} \\
&=\ \frac{(1+\Delta)v(\frac{t_j(1+\Delta)}{1+\nu}) - (1+\nu)^2 v(t_j(1+\nu))}{\Delta - 2\nu - \nu^2}.
\end{aligned}
$$

Again, we combine the bounds:

$$
\begin{aligned}
&\bar{\varphi}(\bar{q}(1+\Delta)(1+\nu)) - \varphi(t_j(1+\nu)) \\
\leq\quad & \frac{(1+\Delta)[\Delta - 2\nu - \nu^2 - \Delta]v(\frac{t_j(1+\Delta)}{1+\nu}) + [\Delta(1+\nu)^2 - (\Delta - 2\nu - \nu^2)]v(t_j(1+\nu))}{\Delta(\Delta - 2\nu - \nu^2)} \\
\leq\quad & \frac{-(2\nu+\nu^2)(1+\Delta)v(\frac{t_j(1+\Delta)}{1+\nu}) + (2\nu+\nu^2)(1+\Delta)v(t_j(1+\nu))}{\Delta(\Delta - 2\nu - \nu^2)} \\
\leq\quad & \frac{(2\nu+\nu^2)(1+\Delta)\left(v(t_j(1+\nu)) - v(\frac{t_j(1+\Delta)}{1+\nu})\right]}{\Delta(\Delta - 2\nu - \nu^2)} \\
\leq\quad & \frac{(2\nu+\nu^2)(1+\Delta)\left[\frac{1+\Delta}{(1+\nu)^2} - 1\right) v(\frac{t_j(1+\Delta)}{1+\nu})}{\Delta(\Delta - 2\nu - \nu^2)} \qquad \text{(by Lemma 6.2, as } 1+\Delta \geq (1+\nu)^2\text{)} \\
\leq\quad & \frac{(1+\nu)(2\nu+\nu^2)(1+\Delta)(\Delta - 2\nu - \nu^2)}{\Delta(\Delta - 2\nu - \nu^2)} v(t_j(1+\Delta)) \qquad \text{(by Lemma 6.2 again)} \\
\leq\quad & \frac{\nu}{\Delta}(2+\nu)(1+\nu)(1+\Delta)v(t_j(1+\Delta)) \;\leq\; 2(1+\nu)^2(1+\Delta)\frac{\nu}{\Delta}\frac{\mathrm{SR}}{t_j(1+\Delta)} \;\leq\; 2(1+\nu)^2\frac{\nu}{\Delta}\frac{\mathrm{SR}}{\bar{q}}.
\end{aligned}
$$

We now show the remaining inequalities. To obtain the first inequality in (i), we note that by Lemma 6.3 and $\mathcal{E}_a$, $q \geq \frac{\bar{q}}{(1+\nu)} > \frac{t_j}{(1+\nu)^2}$, from which the result follows. Similarly, for the second inequality in (ii), $t_j(1+\nu) \geq \bar{q}(1+\nu) \geq q$, and again the result follows. □

6.5 Bounding the Expected Revenue Loss

Finally, we consider an auction with k bidders, where the valuation for the ith bidder comes from regular distribution F_i.

Let $\mathrm{Shtf} = \sum_i E[\varphi_i \cdot x_i] - \sum_i E[\varphi_i \cdot \tilde{x}_i]$. In other words, $\overline{\mathrm{MR}} + \mathrm{Shtf} = \mathrm{MR}$, so it suffices to show that $\mathrm{Shtf} \leq \epsilon \mathrm{MR}$. In the next lemma, we bound Shtf by the sum of three terms, which we will bound in turn. Recall that q_{r_i} denotes the quantile of F_i corresponding to the reserve price for B_i in the Myerson auction and $q_{\bar{r}_i}$ denotes the empirical quantile corresponding to the reserve price in the empirical auction. Also, we let $\bar{q}_i$ be a quantile for B_i in the empirical auction, and we let q_i denote the corresponding quantile in F_i. $\bar{q}_j$ and q_j are defined similarly w.r.t. B_j. Finally, to reduce clutter, we let $\beta = (1+\Delta)(1+\nu)^3 - 1$.

Lemma 6.6.

$$
\begin{aligned}
\mathrm{Shtf} &= \sum_i E\left[\int_{q_i \leq q_{r_i}} \varphi_i(q_i)x_i(q_i)\; dq_i - \int_{q_i \leq q_{\bar{r}_i}} \varphi_i(q_i)\tilde{x}_i(q_i)\; dq_i\right] \\
&\leq \sum_i E\left[\int_{q_i \leq q_{r_i}} \varphi_i(q_i) \cdot [x_i(q_i) - \tilde{x}_i(\frac{q_i}{1+\beta})]\; dq_i \right. && (9) \\
&\qquad + \beta \int_{q_i \leq q_{r_i}} \varphi_i(q_i)\; dq_i && (10) \\
&\qquad \left. + \int_{q_{r_i} \leq q_i \leq q_{\bar{r}_i}} [-\varphi_i(q_i)]\; dq_i. \right]. && (11)
\end{aligned}
$$

Proof. Note that $\tilde{x}_i(q_i) = 0$ for $q_i > q_{\bar{r}_i}$. Thus

$$-\int_{q_i \le q_{\bar{r}_i}} \varphi_i(q_i)\tilde{x}_i(q_i)\ dq = -\int_{q_i \le q_{r_i}} \varphi_i(q_i)\tilde{x}_i(q_i) + \int_{q_{r_i} \le q_i \le q_{\bar{r}_i}} [-\varphi_i(q_i)] \cdot \tilde{x}_i(q_i)\ dq_i$$
$$\le -\int_{q_i \le q_{r_i}/(1+\beta)} \varphi_i(q_i)\tilde{x}_i(q_i) + \int_{q_{r_i} \le q_i \le q_{\bar{r}_i}} [-\varphi_i(q_i)]\ dq_i$$

and

$$-\int_{q_i \le q_{r_i}/(1+\beta)} \varphi_i(q_i)\tilde{x}_i(q_i)\ dq = -(1+\beta)\int_{q_i \le q_{r_i}/(1+\beta)} \varphi_i(q_i)\tilde{x}_i(q_i)\ dq_i + \beta\int_{q_i \le q_{r_i}/(1+\beta)} \varphi_i(q_i)\tilde{x}_i(q_i)\ dq_i$$
$$\le -\int_{q_i \le q_{r_i}} \varphi_i(\frac{q_i}{1+\beta})\tilde{x}_i(\frac{q_i}{1+\beta})\ dq_i + \beta\int_{q_i \le q_{r_i}} \varphi_i(q_i)\tilde{x}_i(q_i)\ dq_i$$
$$\le -\int_{q_i \le q_{r_i}} \varphi_i(q_i)\tilde{x}_i(\frac{q_i}{1+\beta})\ dq_i + \beta\int_{q_i \le q_{r_i}} \varphi_i(q_i)\ dq_i$$

□

In the following lemmas we bound the terms (10), (11), and (9) in turn.

Lemma 6.7.

$$\sum_i \beta\int_{q_i \le q_{r_i}} \varphi_i(q_i)\ dq_i \le k\beta\text{MR}.$$

Proof. Note that $\int_{q_i \le q_{r_i}} \varphi_i(q_i)\ dq_i = \text{SR}_i \le \text{MR}$, by Claim 6.1(ii). The result follows on summing over i. □

We bound (11) by partitioning the integral into two intervals. The intervals are the ranges $q_{r_i} \le q_i \le \max\{\xi_i, q_{r_i}\}$ and $\max\{\xi_i, q_{r_i}\} \le q_i \le q_{\bar{r}_i}$, respectively, where ξ_i is the quantile of F_i corresponding to empirical quantile $\bar{\xi}$.

Lemma 6.8.

$$\sum_i \int_{q_{r_i} \le q_i \le \max\{\xi_i, q_{r_i}\}} [-\varphi_i(q_i)]\ dq_i \le k\bar{\xi}(1+\nu) \cdot \text{MR}.$$

Proof. If $\xi_i \le q_{r_i}$ the integral is zero and the result is immediate. So we can assume that $\xi_i \ge q_{r_i}$. Note that $-\varphi_i(q_i)$ is a non-decreasing function of q_i; thus its smallest values in the range $q_i \ge q_{r_i}$ occur in the integral we are seeking to bound. It follows that

$$\int_{q_{r_i} \le q_i \le \xi_i} [-\varphi_i(q_i)]\ dq_i \le \frac{\xi_i - q_{r_i}}{1 - q_{r_i}} \int_{q_{r_i} \le q_i} [-\varphi_i(q)]\ dq_i$$
$$\le \xi_i \int_{q_{r_i} \le q_i} [-\varphi_i(q_i)]\ dq_i$$
$$\le \xi_i \int_{q_i \le q_{r_i}} \varphi_i(q_i)\ dq_i \qquad \left(\text{as } \int_{0 \le q_i \le 1} \varphi_i(q_i)\ dq_i = 0\right)$$
$$= \xi_i \cdot \text{SR}_i \le \xi_i \cdot \text{MR} \le \bar{\xi}(1+\nu) \cdot \text{MR}.$$

The last two inequalities follow from Claim 6.1(ii) and Lemma 6.3, respectively. The result follows on summing over i. □

We define $\mathcal{E}_b$ to be the event that $\mathcal{E}_a$ holds for every distribution F_i.

Lemma 6.9. *Conditioned on $\mathcal{E}_b$,*

$$E\left[\sum_i \int_{\max\{\xi_i, q_{r_i}\} \le q_i \le q_{\bar{r}_i}} [-\varphi_i(q_i)]\ dq_i\right] \le 2\nu \sum_i \mathrm{SR}_i \le 2k\nu \cdot \mathrm{MR}.$$

Proof. let $\chi_i = \max\{\xi_i, q_{r_i}\}$ and let $\bar{\chi}_i$ be the corresponding empirical quantile. Again, if $\chi_i \ge \bar{q}_{r_i}$ the integral is zero and the result is immediate. So we can assume that $\chi_i < \bar{q}_{r_i}$. The derivation below uses Lemma 6.3 to justify the first and third inequalities, and the second inequality follows from the definition of $\bar{r}_i$ as the empirical reserve price. Conditioned on $\mathcal{E}_a$,

$$q_{\bar{r}_i} \cdot \bar{r}_i \ \ge\ \frac{\bar{q}_{\bar{r}_i} \cdot \bar{r}_i}{1+\nu} \ \ge\ \frac{\bar{\chi}_i \cdot \bar{v}_i(\bar{\chi}_i)}{1+\nu} \ \ge\ \frac{\chi_i \cdot v_i(\chi_i)}{(1+\nu)^2}, \tag{12}$$

Thus

$$\begin{aligned} \int_{\chi_i \le q_i \le q_{\bar{r}_i}} [-\varphi_i(q_i)]\ dq_i \ &\le\ \chi_i \cdot v_i(\chi_i) - q_{\bar{r}_i} \cdot \bar{r}_i \ \le\ \chi_i \cdot v_i(\chi_i)\left[1 - \frac{1}{(1+\nu)^2}\right] \quad \text{(by (12))} \\ &\le\ \frac{\nu(2+\nu)}{(1+\nu)^2}\mathrm{SR}_i \le 2\nu \cdot \mathrm{SR}_i. \end{aligned}$$

□

It remains to bound term (9).

The next lemma bounds the probability that B_i in the empirical auction at quantile $\bar{q}_i/(1+\beta)$ loses by a large amount to B_j at quantile $\bar{q}_j/(1+\beta)$ when B_i at quantile q_i wins against B_j at quantile q_j, and $\bar{q}_j \ge \bar{\xi}$. A large amount is defined as follows: for $\bar{q}_i \ge \bar{\xi}(1+\beta)$ it is more than:

$$2(1+\beta) \cdot \frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta} + 2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta},$$

and for $\bar{q}_i < \bar{\xi}(1+\beta)$ it is more than:

$$2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta}.$$

Lemma 6.10. *Conditioned on $\mathcal{E}_b$, for any pair B_i and B_j, if $\bar{q}_j \ge \bar{\xi}$ then the probability of the following event is bounded by $(1+\beta)^2 - 1$.*
i. If $\bar{q}_i \ge \bar{\xi}(1+\beta)$, the event is

$$\begin{aligned} &\varphi_j(q_j) + 2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta} < \varphi_i(q_i) - 2(1+\beta) \cdot \frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta} \ \textit{and} \\ &\bar{\varphi}_i(\frac{\bar{q}_i}{1+\beta}) < \bar{\varphi}_j(\frac{\bar{q}_j}{1+\beta}). \end{aligned}$$

ii. While if $\bar{q}_i < \bar{\xi}(1+\beta)$, the event is

$$\begin{aligned} &\varphi_j(q_j) + 2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta} < \varphi_i(q_i), \ \textit{and} \\ &\bar{\varphi}_i(\frac{\bar{q}_i}{1+\beta}) < \bar{\varphi}_j(\frac{\bar{q}_j}{1+\beta}). \end{aligned}$$

Proof. We begin with the proof of (i). Given $\mathcal{E}_b$, by Lemma 6.5, for $\bar{q}_i \geq \bar{\xi}(1+\beta)$, $\varphi_i(q_i) \leq \bar{\varphi}_i(\frac{\bar{q}}{1+\beta}) + 2(1+\beta)\cdot\frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta}$ and for $\bar{q}_j \geq \bar{\xi}$, $\bar{\varphi}_j(\bar{q}_j(1+\Delta)(1+\nu)) \leq \varphi_j(q_j) + 2(1+\nu)^2\frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta}$. Thus, if the conditions in (i) hold, then

$$\begin{aligned}\bar{\varphi}_j(\bar{q}_j(1+\Delta)(1+\nu)) &\leq \varphi_j(q_j) + 2(1+\nu)^2\frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta}\\ &< \varphi_i(q_i) - 2(1+\beta)\cdot\frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta}\\ &\leq \bar{\varphi}_i(\frac{\bar{q}_i}{1+\beta})\\ &< \bar{\varphi}_j(\frac{\bar{q}_j}{1+\beta}).\end{aligned}$$

Thus we have a lower bound of $\bar{\varphi}_j(\bar{q}_j(1+\Delta)(1+\nu))$ and an upper bound of $\bar{\varphi}_j(\frac{\bar{q}_j}{1+\beta})$ on the remaining terms. Clearly these can both hold only for a limited range of $\bar{q}_j$ and hence of q_j, which we bound as follows. Define $\hat{\bar{q}}_j = \arg\min_{\bar{q}_j}\{\bar{\varphi}_j(\bar{q}_j(1+\Delta)(1+\nu)) \leq \bar{\varphi}_i(\bar{q}_i/[1+\beta])\}$, and let $\hat{q}_j$ be the corresponding quantile in F_j. Then these bounds can hold for at most $\hat{\bar{q}}_j \leq \bar{q}_j < \hat{\bar{q}}_j(1+\Delta)(1+\nu)(1+\beta)$. To obtain a probability bound, one needs to express the range in terms of the q_j quantile, namely from $\hat{q}_j/(1+\nu)$ to $\min\{1, \hat{q}_j(1+\Delta)(1+\nu)^2(1+\beta)\}$, i.e. with probability at most $(1+\Delta)(1+\nu)^3(1+\beta) - 1 = (1+\beta)^2 - 1$.

To prove (ii), we proceed similarly. In the derivation below, we let $q_i(\bar{q}_i)$ denote the value of q_i corresponding to $\bar{q}_i$.

$$\begin{aligned}\bar{\varphi}_j(\bar{q}_j(1+\Delta)(1+\nu)) &\leq \varphi_j(q_j) + 2(1+\nu)^2\frac{\mathrm{SR}_j}{\bar{q}\,'}\frac{\nu}{\Delta} < \varphi_i(q_i)\\ &\leq \varphi_i(\min\{\xi_i, \frac{q_i}{1+\beta}\})\\ &\leq \bar{\varphi}_i(\frac{q_i}{1+\beta}) < \bar{\varphi}_j(\frac{q_j}{1+\beta})\end{aligned}$$

where the next to last inequality follows because for $\bar{q} \leq \bar{\xi}$, i.e. for $q_i \leq \xi_i$, $\bar{\varphi}_i(\bar{q}_i) = v_i(q_i) \geq \varphi_i(q_i)$. The rest of the argument is as for (i). □

Lemma 6.11. *Conditioned on $\mathcal{E}_b$,*

$$\begin{aligned}&\sum_i \int_{q_i \leq q_{r_i}} \varphi_i(q_i)\cdot[x_i(q_i) - \tilde{x}_i(\frac{q_i}{1+\beta})]\ dq_i\\ &\quad\leq k(k-1)\bar{\xi}(1+\nu)^2\mathrm{MR} + k[(1+\beta)^2-1]\mathrm{MR}\\ &\qquad + 4k(k-1)(1+\beta)(1+\nu)\cdot\ln\frac{1}{\bar{\xi}}\frac{\nu}{\Delta}\mathrm{MR}.\end{aligned}$$

Proof. We introduce notation to measure the probability of wins by small and large margins. Let $x^s_{ij}(q_i, q_j)$ denote the probability that B_i wins in the Myerson auction at quantile q_i, when B_j is at quantile q_j, that $\varphi_i(q_i) - \varphi_j(q_j) \leq 2(1+\nu)^2\frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta} + 2(1+\beta)\cdot\frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta}$ if $\bar{q}_i \geq \bar{\xi}(1+\beta)$, and $\varphi_i(q_i) - \varphi_j(q_j) \leq 2(1+\nu)^2\frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta}$ if $\bar{q}_i < \bar{\xi}(1+\beta)$, and that B_j wins in the empirical auction at quantile $\bar{q}_j/(1+\beta)$, when B_i is at quantile $\bar{q}_i/(1+\beta)$. Similarly, let $\tilde{x}^s_{ij}(q_i, q_j)$ denote the probability that B_i wins in the empirical auction at quantile $\bar{q}_i/(1+\beta)$, when B_j is at quantile $\bar{q}_j/(1+\beta)$, and that B_j wins in the Myerson auction at quantile q_j, when B_i is at quantile q_i, and when $\bar{q}_j \geq \xi(1+\beta)$, that $\varphi_j(q_j) - \varphi_i(q_i) \leq 2(1+\nu)^2\frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta} + 2(1+\beta)\cdot\frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta}$, while if

$\bar{q}_j < \xi(1+\beta)$, then $\varphi_j(q_j) - \varphi_i(q_i) \le 2(1+\nu)^2 \frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta}$. Next we define the probability of a large margin win to be $x_i^l(q_i) = x_i(q_i) - \sum_{j \neq i} \int_{q_j} x_{ij}^s(q_i, q_j)\ dq_j$.

$$\sum_i \int_{q_i \le q_{r_i}} \varphi_i(q_i) \cdot [x_i(q_i) - \tilde{x}_i(\frac{q_i}{1+\beta})]\ dq$$

$$\le \sum_i \int_{q_i \le q_{r_i}} \varphi_i(q_i) \cdot x_i^l(q_i)\ dq_i \tag{13}$$

$$+ \sum_i \sum_{j \neq i} \int_{q_i \le q_{r_i}, q_j \le \xi_j(1+\nu)} \phi_i(q_i)\ dq_i\ dq_j \tag{14}$$

$$+ \sum_i \sum_{j \neq i} \Bigg[\int_{q_i \le q_{r_i}, \xi_j(1+\nu) \le q_j \le q_{r_j}} \varphi_i(q_i) \cdot x_{ij}^s(q_i, q_j)\ dq_i\ dq_j \tag{15}$$

$$- \int_{q_i \le q_{r_i}, q_j \le q_{r_j}} \varphi_i(q_i) \cdot \tilde{x}_{ij}^s(q_i, q_j)\ dq_i\ dq_j \Bigg] \tag{16}$$

We bound (13)–(14) in turn.

By Lemma 6.10, $x_i^l(q_i) \le (1+\beta)^2 - 1$. Thus

$$\begin{aligned}
\sum_i \int_{q_i \le q_{r_i}} \varphi_i(q_i) \cdot x_i^l(q_i)\ dq &\le [(1+\beta)^2 - 1] \sum_i \int_{q_i \le q_{r_i}} \varphi_i(q_i)\ dq_i \\
&\le [(1+\beta)^2 - 1] \sum_i \mathrm{SR}_i \\
&\le k[(1+\beta)^2 - 1]\mathrm{MR}.
\end{aligned}$$

$$\begin{aligned}
\sum_i \sum_{j \neq i} \int_{q_i \le q_{r_i}, q_j \le \xi_j(1+\nu)} \phi_i(q_i)\ dq_i\ dq_j &\le \sum_i \sum_{j \neq i} (1+\nu)\bar{\xi}_j \mathrm{SR}_i \\
&\le k(k-1)\bar{\xi}(1+\nu)^2\mathrm{MR} \quad \text{(as } \xi_j \le \bar{\xi}(1+\nu)\text{)}.
\end{aligned}$$

Note that $x_{ij}^s(q_i, q_j) = \tilde{x}_{ji}^s(q_i, q_j)$. Thus when $\bar{q}_i \ge \bar{\xi}(1+\beta)$, $\varphi_i(q_i) \cdot x_{ij}^s(q_i, q_j) - \varphi_j \cdot \tilde{x}_{ij}^s(q_j . q_i) \le x_{ij}^s[2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta} + 2(1+\beta) \cdot \frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta}]$, and if $\bar{q}_i < \bar{\xi}(1+\beta)$, the bound is $x_{ij}^s[2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta}]$.

We conclude that

$$\sum_i \sum_{j\neq i} \left[\int_{q_i \le q_{r_i}, \xi_j(1+\beta)\le q_j \le q_{r_j}} \varphi_i(q_i) \cdot x^s_{ij}(q_i,q_j)\ dq_i\ dq_j - \int_{q_i\le q_{r_i}, q_j \le q_{r_j}} \varphi_i(q_i)\cdot \tilde{x}^s_{ij}(q_i,q_j)]\ dq_i\ dq_j \right]$$

$$\le \sum_i \int_{\xi_i(1+\nu)<q_i\le q_{r_i}, \xi_j(1+\beta)\le q_j\le q_{r_j}} \sum_{j\neq i} 2(1+\nu)^2 \frac{\mathrm{SR}_j}{\bar{q}_j}\frac{\nu}{\Delta} + 2(1+\beta)\cdot \frac{\mathrm{SR}_i}{\bar{q}_i}\frac{\nu}{\Delta}\ dq_i\ dq_j$$

$$+ \sum_i \int_{q_i \le \xi_i(1+\nu), \xi_j(1+\beta)\le q_j \le q_{r_j}} \sum_{j\neq i} 2\frac{\mathrm{SR}_j(1+\nu)^2}{\bar{q}_j}\frac{\nu}{\Delta}\ dq_i\ dq_j$$

$$\le \sum_i \int_{q_i\le q_{r_i}, \xi_j(1+\nu)\le q_j\le q_{r_j}} \sum_{j\neq i} 2\frac{\mathrm{SR}_j(1+\nu)^3}{q_j}\frac{\nu}{\Delta}\ dq_i\ dq_j \quad (\text{as } \bar{q}_j \ge \frac{q_j}{1+\nu})$$

$$+ \sum_i \int_{\xi_i(1+\nu)\le q_i\le q_{r_i}} 2[q_{r_j} - \xi_j(1+\beta)](1+\beta)\cdot \frac{\mathrm{SR}_i(1+\nu)}{q_i}\frac{\nu}{\Delta}\ dq_i$$

$$\le \sum_i \sum_{j\neq i} \left[2\cdot \mathrm{SR}_j(1+\nu)^3 \ln\frac{1}{\xi_j(1+\nu)}\frac{\nu}{\Delta} + 2(1+\beta)(1+\nu)\cdot \mathrm{SR}_i \ln \frac{1}{\xi_j(1+\beta)}\frac{\nu}{\Delta}\right]$$

$$\le 4k(k-1)(1+\beta)(1+\nu)\cdot \ln\frac{1}{\bar{\xi}}\frac{\nu}{\Delta}\mathrm{MR} \quad (\text{since } \xi_i,\xi_j \ge \frac{\bar{\xi}}{1+\nu} \ge \frac{\bar{\xi}}{1+\beta}).$$

□

We are now ready to bound Shtf.

Lemma 6.12.

$$\mathrm{Shtf} \le \mathrm{MR}\left[k(k+1)\delta + k\beta + 2k\nu + k^2\bar{\xi}(1+\nu)^2 + k[(1+\beta)^2-1]\right] + \mathrm{MR}\left[4k(k-1)(1+\beta)(1+\nu)\cdot \ln\frac{1}{\bar{\xi}}\frac{\nu}{\Delta}\right].$$

Proof. In the event that $\mathcal{E}_a$ does not hold for some F_i, which occurs with probability at most $k\delta$, the contribution to Shtf is at most

$$k\delta\left[\sum_i \int_{q_i\le q_{r_i}} \varphi_i(q_i)x_i(q_i)\ dq_i - \int_{q_i\le \bar{q}_{r_i}} \varphi_i(q_i)\tilde{x}_i(q_i)\ dq_i\right]$$

$$\le k\delta\mathrm{MR} + k\delta\sum_i \int_{q_{r_i}<q_i\le \bar{q}_{r_i}} [-\varphi_i(q_i)]\ dq_i$$

$$\le k\delta\mathrm{MR} + k\delta\sum_i \int_{q_i\le q_{r_i}} \varphi_i\ dq_i$$

$$\le k\delta\mathrm{MR} + k\delta\cdot k\mathrm{MR} = k(k+1)\mathrm{MR}.$$

Otherwise, the contribution is given by summing the bounds from Lemmas 6.7–6.9 and 6.11. □

Proof of Theorem 3.1: We first choose $\Delta, \nu \le \frac{1}{12}$. It is easy to check that then $(1+\beta)^2 - 1 = (1+\Delta)^2(1+\nu)^6 - 1 = 2\Delta(1+\Delta)(1+\nu)^6 + 6\nu(1+\Delta)^2(1+\nu)^5 \le (2\Delta+6\nu)\left(\frac{13}{12}\right)^7 \le 4\Delta + 11\nu$, and similarly $\beta = (1+\Delta)(1+\nu)^3 - 1 \le 2\Delta + 4\nu$. Thus $3k\nu + k\beta + k[(1+\beta)^2-1] \le 6k\Delta + 18k\nu$. Finally, $4(1+\beta)(1+\nu) \le 4\left(\frac{13}{12}\right)^5 \le 4\cdot\frac{3}{2} = 6$. Consequently,

$$\mathrm{Shtf} \le \mathrm{MR}\left[k\delta + 6k\Delta + 2k^2\bar{\xi} + 18\nu + 6k(k-1)\ln\frac{1}{\bar{\xi}}\frac{\nu}{\Delta}\right].$$

It suffices that Shtf $\leq \epsilon$MR. To this end, we bound the right hand side of the above expression by ϵ. To achieve this it suffices to choose ν, $\bar{\xi}$, δ, and Δ as follows:

$$\begin{aligned} k(k+1)\delta &= \frac{1}{4}\epsilon \\ 6k\Delta &= \frac{1}{4}\epsilon \\ 2k^2\bar{\xi} &= \frac{1}{4}\epsilon \\ 18k\nu + 6k(k-1)\ln\frac{1}{\bar{\xi}}\frac{\nu}{\Delta} &= \frac{1}{4}\epsilon. \end{aligned}$$

It suffices that

$$\begin{aligned} \delta &= \Theta\left(\frac{\epsilon}{k^2}\right) \\ \Delta &= \Theta\left(\frac{\epsilon}{k}\right) \\ \bar{\xi} &= \Theta\left(\frac{\epsilon}{k^2}\right) \\ \nu &= \Theta\left(\frac{\epsilon^2}{k^3}\frac{1}{\ln k + \ln\frac{1}{\epsilon}}\right). \end{aligned}$$

One final detail is that we need to set $\hat{\xi}$ also, but it suffices to note that $\hat{\xi} = \bar{\xi} + \frac{1}{2m}$ and so $\hat{\xi} = \Theta(\frac{\epsilon}{k^2}) + \frac{1}{2m}$.

By Lemma 6.3, $m = \Omega(\frac{1}{\gamma^3\hat{\xi}} + \ln\frac{1}{\delta}\cdot\frac{1}{\gamma^2\hat{\xi}})$ suffices. Recalling that $1+\nu = (1+\gamma)^2$, so $\gamma = \Theta(\nu)$, we obtain that $m = \Omega(\frac{k^{11}}{\epsilon^7}(\ln^3 k + \ln^3\frac{1}{\epsilon})$ suffices. □

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