Fast-Converging Tatonnement Algorithms for One-Time and Ongoing Market Problems *

Richard Cole
Computer Science Department
Courant Institute of Mathematical Sciences
New York University
New York, NY 10012-1185, USA
cole @cs .nyu .edu

Lisa Fleischer
Department of Computer Science
Dartmouth College
Hanover, NH 03755, USA
lkf @cs .dartmouth .edu

ABSTRACT

Why might markets tend toward and remain near equilibrium prices? In an effort to shed light on this question from an algorithmic perspective, this paper formalizes the setting of *Ongoing Markets*, by contrast with the classic market scenario, which we term *One-Time Markets*. The Ongoing Market allows trade at non-equilibrium prices, and, as its name suggests, continues over time. As such, it appears to be a more plausible model of actual markets.

For both market settings, this paper defines and analyzes variants of a simple tatonnement algorithm that differs from previous algorithms that have been subject to asymptotic analysis in three significant respects: the price update for a good depends only on the price, demand, and supply for that good, and on no other information; the price update for each good occurs distributively and asynchronously; the algorithms work (and the analyses hold) from an arbitrary starting point.

Our algorithm introduces a new and natural update rule. We show that this update rule leads to fast convergence toward equilibrium prices in a broad class of markets that satisfy the weak gross substitutes property. These are the first analyses for computationally and informationally distributed algorithms that demonstrate polynomial convergence.

Our analysis identifies three parameters characterizing the markets, which govern the rate of convergence of our protocols. These parameters are, broadly speaking:

1. A bound on the fractional rate of change of demand for each good with respect to fractional changes in its price.
2. A bound on the fractional rate of change of demand for each good with respect to fractional changes in wealth.
3. The closeness of the market to a Fisher market (a market with buyers starting with money alone).

We give two types of protocols. The first type assumes global knowledge of only (an upper bound on) the first parameter. For this protocol, we also provide a matching lower bound in terms of these parameters for the One-Time Market. Our second protocol, which is analyzed for the One-Time Market only, assumes no global knowledge whatsoever.

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1. INTRODUCTION

The impetus for this work comes from the following question: why are well-functioning markets able to stay at or near equilibrium prices?[1] This raises two issues: what are plausible price adjustment mechanisms and in what types of markets are they effective?

This question was originally considered by Walras in 1874, when he suggested that prices adjust by tatonnement: upward if there is too much demand and downward if too little [30]. Since then, the study of market equilibria, their existence, stability, and their computation has received much attention in Economics, Operations Research, and most recently in Computer Science. Of late, this has led to a considerable number of polynomial time algorithms for finding approximate and exact equilibria in a variety of markets with divisible goods. However, these algorithms do not seek to, and do not appear to provide methods that might plausibly explain these markets' behavior.

We argue here for the relevance of this question from a computer science perspective. Much justification for looking at the market problem in computer science stems from the following argument: If economic models and statements about equilibrium and convergence are to make sense as being realizable in economies, then they should be concepts that are computationally tractable. Our viewpoint is that it is not enough to show that the problems are computationally tractable; it is also necessary to show that they are tractable in a model that might capture how a market works. Unless one has a controlled economy, markets surely do not perform overt global computations, using global information.

In formalizing the tatonnement model, economists have proposed models to capture aspects of how a market might work; and convergence of several of these formalizations has been demonstrated for

[1] We are not concerned with the question of whether this assertion is indeed correct.

some types of markets [1, 2, 21, 27]. However, there is no demonstration that these proposed models converge reasonably quickly. Indeed, without care in the specific details, they won't.[2]

At first sight, it is not clear why these models are realistic. The most studied, due to Walras, is the auctioneer model: an auctioneer announces prices, receives the market demand at these prices in the form of buy and sell requests, *but with no trade actually occurring*, adjusts prices according to the tatonnement procedure, and iterates. Only when prices reach equilibrium is trade allowed. In reality, it is trade that reveals demand and hence needed price adjustments. Thus any realistic model has to enable trade in disequilibrium.

In this paper we propose a simple market model in which the market extends over time and trading occurs out of equilibrium (as well as at equilibrium). We call this the *Ongoing Market*. Here, the market repeats from one time unit to the next; we call the basic unit a *day*. The link from one day to the next is that goods unsold one day are available the next day, in addition to the new supply, which for simplicity, we take as being the same from day to day. This appears to provide a simple and natural way of allowing out-of-equilibrium trade. The algorithmic task is to converge to equilibrium prices while clearing unsold stocks. We develop a novel tatonnement algorithm for this model and show that it results in rapid convergence toward equilibrium prices in this market.

Our analysis of the algorithm for the Ongoing Market relies on new tatonnement algorithms and understanding which we develop by analyzing the more traditional market problem discussed in the paragraph preceding the last. In this paper, we call this the *One-Time Market*. The algorithmic technique of iteratively computing prices for the One-Time Market can be seen as a plausible approximation to the Ongoing Market (but with no carry over of unsold goods). Our work can be seen as a formal justification for this approach, as well as a validation of Walras' intuition regarding tatonnement. Further, in our opinion, the intuitive understanding that markets are usually similar from one time period to the next has been a factor in the previous appeal of iterative price update algorithms, including, in the Computer Science literature, the recent tatonnement algorithm of Codenotti et al. [5] and the auction algorithms of Garg et al. [12].

Our proposed price update protocols capture important characteristics of trading as proposed in the economic literature, features that are lacking from previous algorithms subject to asymptotic analysis. Namely, our algorithms consist of price updates satisfying the following three criteria: the price update for a good depends only on the price, demand, and supply for that good, and on no other information about the market; the price update for each good occurs distributively and asynchronously; the algorithms can start with an arbitrary set of prices. We show that our update protocols converge quickly in many markets that satisfy the weak gross substitutes property. In the process, we identify three natural parameters characterizing markets that govern the rate of convergence.

1.1 The Market Problems

The One-Time Market[3] A market comprises two sets, goods G, with $|G| = n$, and agents A, with $|A| = m$. The goods are assumed to be infinitely divisible. Each agent l starts with an allocation w_{il} of good i. Each agent l has a utility function $u_l(x_{1l}, \cdots, x_{nl})$ expressing its preferences: if l prefers a basket with x_{il} units (possibly a real number) of good i, to the basket with y_{il} units, for $1 \le i \le n$, then $u_l(x_{1l}, \cdots, x_{nl}) > u_l(y_{1l}, \cdots, y_{nl})$. Each agent l intends to trade goods so as to achieve a personal optimal combination (basket) of goods given the constraints imposed by their initial allocation. The trade is driven by a collection of prices p_i for good i, $1 \le i \le n$. Agent l chooses x_{il}, $1 \le i \le n$, so as to maximize u_l, subject to the basket being affordable, that is: $\sum_{i=1}^n x_{il}p_i \le \sum_{i=1}^n w_{il}p_i$. Prices $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ are said to provide an *equilibrium* if, in addition, the demand for each good is bounded by the supply: $\sum_{j=1}^m x_{il} \le \sum_{l=1}^m w_{il}$. The market problem is to find equilibrium prices.[4]

Standard notation $w_i = \sum_l w_{il}$ is the supply of good i. $x_i = \sum_l x_{il}$ is the demand for good i, and $z_i = x_i - w_i$ is the excess demand for good i (which can be positive or negative). $v_l(\mathbf{p}) = \sum_i w_{il}p_i$ is the wealth of buyer l given prices $\mathbf{p}$. Where $\mathbf{p}$ is clear from context, we express wealth simply as v_l. Note that while w is part of the specification of the market, v, x and z are functions of the vector of prices: v directly so, and x and z as determined by individual agents maximizing their utility functions subject to v.

We follow standard practice[5] and view the actions of individual buyers and sellers as being encapsulated in the price adjustments for each good. More specifically, we imagine that there is a separate, "virtual" price setter for each good in the market. Henceforth, for ease of exposition, we describe price setters as if they were actual entities, although in reality they are virtual entities induced by agents' trades.

The Ongoing Market In order to have non-equilibrium trade, we need a way to allocate excess supply and demand. To this end, we suppose that for each good there is a *warehouse* which can store excess demand and meet excess supply. This is most readily analyzed in the *Fisher market* setting, a special case of the exchange market in which buyers and sellers are distinct. In a Fisher market, a buyer's initial allocation is just money and its desire is to purchase non-money goods. A seller's allocation is a single good and its desire is for money alone. Without loss of generality, there is a single seller for each good, who is therefore the price setter for that good. The seller has a warehouse of finite capacity to enable it to cope with fluctuations in demand. It will change prices as needed to ensure its warehouse neither overfills nor runs out of goods.

The market consists of a set G of n goods and a set B of m buyers. The market repeats over an unbounded number of time intervals called days. Each day, seller of good i (called seller i) receives w_i new units of good i, and buyer ℓ is given v_ℓ money, $1 \le \ell \le m$. As before, each buyer ℓ has a utility function $u_\ell(x_{1\ell}, \cdots, x_{n\ell})$ expressing its preferences. Each day, buyer ℓ selects a maximum utility basket of goods $(x_{1\ell}, \cdots, x_{n\ell})$ of cost at most v_ℓ. Each seller i provides the demanded goods $\sum_{\ell=1}^m x_{i\ell}$. The resulting excess demand or surplus, $\sum_{\ell=1}^m x_{i\ell} - w_i$, is taken from or added to the warehouse stock. Seller i has a warehouse of capacity c_i.

Given initial prices p_i°, warehouse stocks s_i°, where $0 < s_i^\circ < c_i$, $1 \le i \le n$, and ideal warehouse stocks s_i^F, $0 < s_i^F < c_i$, the task is to repeatedly adjust prices so as to converge to equilibrium prices with the warehouse stocks converging to their ideal values. We let s_i denote the current contents of warehouse i, and $h_i = s_i - s_i^F$ denote the *excess warehouse reserves*.

The difficulty with the problem as stated is that the initial prices could be arbitrarily low and hence demand arbitrarily high, thereby causing the seller(s) to run out of stock. To avoid this, we allow

[2]Of the referenced papers, only one formalization [27] is a discrete algorithm, and, as we make more specific later, it may not converge quickly.

[3]The market we describe here is often referred to as the *exchange* market or *Arrow-Debreu* market. We use a different term because we consider this problem in a new computational model as described in Section 1.2

[4]Equilibria exist under quite mild conditions (see [19] §17.C, for example).

[5]See Varian [28] §21.5.

sellers to change prices sufficiently often. This entails measuring demand on a finer scale than day units. We take a very simple approach: we assume that each buyer spends their money at a uniform rate throughout the day. (Equivalently, this is saying that buyers with collectively identical profiles occur throughout the day, though really similar profiles suffice for our analysis.) Likewise, if one supposes there is a limit to the granularity, this imposes a limit on how extreme the initial prices can be for convergence to be assured.

Market Properties In an effort to capture the distributed nature of markets and the likely limited computational power of individual interactions and consequently of each of the virtual price setters, we impose several constraints on procedures we wish to consider:

1. Limited information: the (virtual) price setter for good i knows only the price, supply, and excess demand of good i, both current and past history. Thus the price updates can depend on this information only. Notably, this precludes the use not only of other prices or demands, but also of any information about the specific form of utility functions.

2. Simple actions: The price setters' procedures should be simple.

3. Asynchrony: Price updates for different goods are allowed to be asynchronous.

4. Fast Convergence: The price update procedure should converge quickly toward equilibrium prices from any initial price vector.

We call procedures that satisfy the first three constraints *local*, by contrast with centralized procedures that use more complete (global) information about the market.

Next, we discuss the motivations for these constraints.

Constraint (1) stems from the plausible assertion that not everything about the market will be known to a single price setter. While no doubt some information about several goods is known to a price setter, it is a conservative assumption to assume less is known, for any convergence result carries over to the broader setting. Further, it is far from clear how to model the broader setting.

Constraint (2), simplicity, is in the eye of the beholder. Its presence reflects our view that without further information, this is both generally applicable and plausible.

Constraint (3), asynchrony, is an inherent property of independent price adjustments. Since the price setter of good i reacts only to trade in good i, the price adjustment of good i occurs independently of other price adjustments.

Constraint (4) arises in an effort to recognize the dynamic nature of real markets, which are subject to changing supplies and demands over time. However, surely much of the time, markets are changing gradually, for otherwise there would be no predictability. A natural approximation is to imagine fixed conditions and seek to come close to an equilibrium in the time they prevail — hence the desire for rapid convergence.

1.2 Our Contribution

We describe new and natural *local* price update protocols that converge quickly toward equilibrium prices starting from arbitrary initial prices: the longer they run, the closer they come. To specify this more precisely we need to define the computational model, our complexity measure, and our measure for approximation quality.

Computational model Since we are proposing a model for how a market might reach equilibrium, instead of how one might compute an equilibrium given all the information about the market, our computational model is a bit different from the standard computer science model. Our model is based on iterations, defined below.

Iteration r:

1. **Price updates** Simultaneously for each good i in some subset G^r of goods, the price setter for good i updates the price of good i using knowledge only of p_i, z_i, and the history of p_i and z_i.

2. **Demand updates** Given new prices $\mathbf{p}^r$, agents compute the wealth they could achieve by selling all their goods. Ideally, agents express their interest in a set of goods that maximizes their utility subject to their current wealth. We relax this requirement by allowing aggregate demand to depart from this optimal value as follows: for an input parameter $\sigma > 0$, realized demand $y_i(\mathbf{p})$ for good i satisfies $\frac{1}{1+\sigma} \le \frac{y_i(\mathbf{p})}{x_i} \le 1+\sigma$, where $x_i(\mathbf{p})$ is the demand if each agent maximizes his or her utility. This allows for suboptimal behavior on the part of the agents, as well as for a non-exact aggregation process. We call this the behavior of *σ-approximate optimizers*. Regardless of whether we assume exact or approximate optimization by the agents, utility functions are revealed only implicitly and partially through the aggregate demands for goods subject to a price vector.

It might seem more natural that the price setter for a good i that has not updated the price in the previous iteration would use an old value of the excess demand, or some convex combination of the excess demands seen since the last price update. Our analysis works for any of these variants. This provides further evidence that the update procedure proposed here is robust.

Complexity measure In the One-Time market, as is standard for asynchronous algorithms, we measure the complexity in rounds. The basic unit of time is a price update iteration as specified above. A *round* comprises a minimum length sequence of iterations in which every price updates at least once. The rounds are specified uniquely by defining them beginning from a fixed start time.

In the Ongoing Market, we will require each price to be updated at least once a day, and then a day has the same role as the round in the One-Time market.[6]

b-Bounded asynchrony Sometimes it is useful to limit the extent of the asynchrony. We define b-bounded asynchrony to impose the requirement that in a single round any price updates at most b times.

Approximation quality The main approach in the Computer Science literature has been to define the quality of an allocation x as $1-\epsilon = \min_{l\in A}\{u_l(x_l)/u_l(Opt_l)\}$, where Opt_l is agent l's preferred affordable allocation at the prevailing prices. This does not seem a feasible approach in our setting, where no allocation mechanism is specified, where there is no direct knowledge of the agents' utilities, and our algorithms are just responding to excess demands and not to the degree to which agents wish to change their allocations. More generally, the dependence of the approximation criteria on u could be viewed as problematic: The role of u is to describe a preference order on allocations. Different u give the same preference order, but widely varying approximation guarantees according to the above measure. Instead, we simply measure the distance

[6]This requirement is only necessary if we are interested in the convergence of prices to the equilibrium of a market that includes all the goods. We could omit some goods from this daily update requirement, but then the best we could hope for is that the market converges to an equilibrium on a submarket of the remaining goods and remaining funds not spent on the excluded goods.

from the equilibrium prices, p_i^*, directly: $\max_i |p_i^* - p_i|/p_i^*$.[7]

Our update algorithms For the One-Time market, we analyze protocols where price setters use the rule

$$p_i \leftarrow p_i(1 + \lambda_i \min\{1, z_i/w_i\}). \quad (1)$$

The price of money remains at one. This is a new price update rule. It differs significantly from the update suggested by Uzawa [27] in that it scales a bounded excess demand by the current price. These differences are crucial for enabling a proof of rapid convergence. In particular, the min term prevents overreaction to large values of z_i; these can be unbounded in their effect in Uzawa's algorithm. The scaling by p_i can also improve the rate of convergence significantly.

For the Ongoing Market, we need to clear the warehouse excess; however, if we try to do it all in one day, this will cause prices to overreact. Instead, we use a target demand of $\widetilde{x}_i = w_i + \kappa_i h_i$, where κ_i is a suitable parameter to be specified. We define $\widetilde{z}_i = x_i - \widetilde{x}_i$. Here, our price setters use the rule:

$$p_i \leftarrow p_i(1 + \lambda_i \min\{1, \tilde{z}_i/w_i\}). \quad (2)$$

We begin by analyzing the protocol in the One-Time market when λ_i is *fixed* for all goods i, and given by a simple characteristic of the market. Our motivation for this is two-fold. First, in stable markets, it seems reasonable that the appropriate step sizes for the price adjustments are known (within constant factors). Second, such an analysis has not been done before, and it is helpful to understand this case first.

Then, again for the One-Time market, we analyze an *oblivious* protocol where the appropriate choice of λ_i is not known at the start, and is therefore repeatedly adjusted to ensure that it is eventually small enough for convergence.

Finally, we analyze the non-oblivious protocol in the Ongoing Market.

The performance of all algorithms depends on several global parameters of the market. These relate to how effectively surpluses and scarcities signal the level of price inequities. The parameters are denoted by α, β, E, where $\alpha, \beta \le 1$ and $E \ge 1$, and are discussed in further detail in Section 2.1.

In suitable markets (specified in Section 3), we show that in One-Time markets our first algorithm improves the accuracy of the least accurate price by at least one bit in $O(E/(\alpha\beta))$ rounds. (For small prices, this means doubling the price; for large ones, this means halving it; and for prices p_i close to equilibrium price p_i^*, this means halving $|p_i^* - p_i|$.) We can show that we show that there are examples for which this complexity bound is tight for our update procedures. In Ongoing Markets, the performance is analogous, but the improvement is to whichever is worse, the warehouse excess or the price imbalance, and it requires $O((E/\beta)2)$ rounds for a one-bit improvement (in the Fisher market setting, $\alpha = 1$).

Our second algorithm is *oblivious* in that it does not assume that a convergent value for λ_i is known. Instead the protocol for good i gradually reduces the value of λ_i. To obtain a complexity bound, we need to assume b-bounded asynchrony, and then we obtain roughly the square of the above complexity. Specifically, a d-bit improvement in accuracy takes $O(\frac{b}{\alpha 2\beta 2}(E2 + d2))$ rounds.

1.3 Paper Organization

In Section 2 we give some relevant definitions and describe the parameters. In Section 3 we specify our protocols and results. Section 4 discusses prior work. In Section 5 we note some open questions. In Section 6 we sketch a proof of an upper bound on the rate of convergence of the fixed protocol for the One-Time and ongoing settings in Fisher markets ($\alpha = 1$) with parameters β, E.

[7]One might be tempted to argue that one should measure the quality of an approximate equilibria in terms of the excess demands rather than the error in the prices, but this will have no effect on the rate of convergence, although it can change the percentage error.

2. DEFINITIONS AND NOTATION

A market satisfies the *gross substitutes property* if for any good i, increasing p_i leads to increased demand for all other goods. The market satisfies *weak gross substitutes* if the demand for every other good increases or stays the same. Examples of markets that may satisfy the gross substitutes property include markets of raw materials, energy, airline seats, toll roads. A broad enough market will not satisfy this property. Consider, for example, the market for bread and jam.

Next, we state some common concepts/assumptions regarding the market problem.

Walras' Law:[8] For any price vector $\mathbf{p}$, $\sum_{i \in G} z_i(\mathbf{p})p_i = 0$.

Homogeneity of degree 0:[9] For all price vectors $\mathbf{p}$ and scalars $a > 0$, $x(\mathbf{p}) = x(a\mathbf{p})$.

Numeraire: Under the assumption of homogeneity, if there is at least one equilibrium price vector, then there is an entire ray of equilibria. It is convenient to use normalization to remove this duplication. A common form of normalization used in the economics literature is the concept of the numeraire: choose one good as the *numeraire*; set its price to 1; scale all other prices accordingly.[10] We use money as the numeraire, and use the index \$ to denote this good. Usually the choice of a good to be the numeraire is viewed as arbitrary. However, as we will see, the rate of convergence of our algorithms also depends on how pervasively the numeraire is present throughout the market, and consequently we do not view it as an arbitrary choice.

Uniqueness of Equilibria: It is well-known that under normalization, markets of gross substitutes have a unique equilibrium.[11] Since we focus on markets satisfying gross substitutes, the markets we consider have a unique equilibrium. Throughout the paper we will use the superscript * to denote a characteristic of an equilibrium. For example, $\mathbf{p}^*$ is the equilibrium price vector; $\mathbf{x}^*$ is the equilibrium demand.

2.1 The Parameters

Here, we define the three parameters E, α, β appearing in our analysis.

Elasticity of Demand and the Parameter E: The *price elasticity of demand* is the fractional rate of change of demand with respect to price: $\frac{\partial x_i/\partial p_i}{x_i/p_i}$.[12] Under the assumption of gross substitutes, this is negative. The parameter E is an upper bound on the absolute value of this quantity over all goods and all prices:

$$E = -\min_{i,\mathbf{p}} \frac{\partial x_i/\partial p_i}{x_i/p_i}$$

In markets of weak gross substitutes, $E \ge 1$. In general E could be unbounded (e.g., when utility functions are linear). Intuitively, it is clear that the larger E is, the smaller the price adjustments should be for a given level of excess demand; as a result λ needs to be

[8]See Mas-Colell [19], page 23.
[9]Ibid.
[10]Ibid., page 24.
[11]Ibid., page 613.
[12]Ibid., page 27.

chosen correspondingly small enough so that adjustments ensure convergence. Were the value of the fractional derivative consistent for all prices this would not matter. However, when E is large, $(\partial x_i/\partial p_i)/(x_i/p_i)$ cannot be large for all prices and goods[13]. The outcome is an $O(1/E)$ convergence speed.

Similarly, it would be reasonable to expect a lower bound on the elasticity of demand to affect the convergence rate, and indeed it does. However, if we simply bound this value, by say β, $0 < \beta \leq 1$, analogous to the upper bound E, the rate of convergence appears to depend linearly on $|G|$, the number of goods. To avoid this we make a stronger assumption.

Normal Goods and the Parameter β: Good i is said to be *normal* for agent l if $\partial x_{il}(\mathbf{p}, v_l)/\partial v_l \geq 0$, where here v_l is the wealth of agent l at prices $\mathbf{p}$.[14] We impose the slightly stronger constraint that states for all $(\mathbf{p}, v_l(\mathbf{p}))$ there is a $\beta > 0$ such that $\partial x_{il}(\mathbf{p}, v_l(\mathbf{p}))/\partial v_l(\mathbf{p}) \geq \beta x_{il}/v_l$. In words, it says that the fractional rate of change of demand with respect to wealth is lower bounded by a strictly positive value. We call this the *wealth effect*. More precisely, we define the parameter $0 < \beta$ as

$$\beta = \min_{l,i} \frac{\partial x_{il}(\mathbf{p}, v_l)/\partial v_l}{x_{il}/v_l}.$$

In markets of weak gross substitutes, $\beta \leq 1$. We conjecture that the following alternate assumption leads to the same convergence rate, again independent of $|G|$: β is simply a lower bound on the elasticity and in addition each agent desires $O(1)$ goods. (However, this appears to need a different analysis.)

The Numeraire and the Parameter α: A separate parameter α_i is defined for each good i. To calculate α_i, determine the wealth used to purchase good i; α_i is the fraction of this wealth that came from the initial allocation of money (the numeraire), evaluated at equilibrium prices. Formally,

$$\alpha_i = \frac{1}{w_i} \sum_l \frac{x^*_{il} w_{\$l}}{v_l(p^*)}.$$

We then define $\alpha = \min_i \alpha_i$.

To see why the α could have a natural effect on the convergence rate in markets with a numeraire (such as money), consider the following example market: a market in which there is no allocation of money. Then doubling all prices (given homogeneity of degree zero) would have no effect on demand. It is now plausible, and turns out to be the case, that if only very little money is present in the market (i.e. at equilibrium, the value of the money is very small compared to that of the other goods), then the effect of price changes on demand is muted (or viewed inversely, even if the prices are quite far from equilibrium, the excess demands, and hence the signal they provide, are small).

We note that in markets with CES utilities[15], E is a parameter of the utility (sometimes denoted s) and $\beta = 1$, while in markets with Cobb-Douglas utilities $E = 1$ and $\beta = 1$.

[13]For if $E > 1$ for all prices and goods, imagine starting at equilibrium prices and then reducing the price for one good; eventually all interested buyers would be purchasing only that good; any further price reductions would induce a rate of change of demand for that good with $E \leq 1$.

[14]See Mas-Colell [19], p. 25. This is considered a reasonable constraint for broad categories of goods, such as "food": i.e. as wealth increases, spending on food generally increases, although spending on specific types of food may decrease.

[15]ibid., p. 97.

3. PROTOCOLS AND RESULTS

Our convergence result depends on a natural, but slightly technical, notion of distance of a price to the equilibrium price. We define the *distance* between prices p_i and p^*_i to be $\frac{p^*_i}{p_i}$ if $p^*_i \geq 3p_i$, $\frac{p^*_i - p_i}{p^*_i}$ if $p_i \leq p^*_i < 3p_i$, and $\frac{p_i - p^*_i}{p^*_i}$ if $p^*_i < p_i$.[16] The motivation for this definition is that if there is a big gap between p_i and p^*_i, then our goal is to reduce the ratio, while if p_i is close to p^*_i, then our goal is to reduce their difference. We let $\eta(p_i)$ denote this distance for good i, and $\eta(\mathbf{p}) = \max_i \eta(p_i)$.

The ideal case for our update rule occurs in the One-Time Fisher Market where all buyers have Cobb-Douglas utilities. That is, each buyer wants to spend preset fractions of its wealth on specified goods (e.g. 5% on Good 1, 15% on Good 2, etc. for Buyer 1, 10% on Good 1 etc. for Buyer 2, ...). In this setting $\alpha = \beta = E = 1$, and if each $\lambda_i = 1$, our protocol converges in one step.

In general, $0 \leq \alpha, \beta \leq 1$, and $E \geq 1$. The values of these parameters indicate the divergence of the market from the ideal case, and the greater the divergence, the slower the convergence. We have already discussed why the performance of tatonnement algorithms are likely to depend on E and β, and we have lower bound examples showing that the analysis in the one-time market is tight with respect to all three parameters simultaneously.

One-Time Market Results

THEOREM 3.1 (UPPER BOUND). *In the One-Time Market, the price update protocol given by (1), in weak gross substitutes markets with parameters α, β, E, and initial prices $\mathbf{p}^\circ > 0$, yields price vector $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$ in $O(\frac{1}{\alpha\beta\lambda}(\log \frac{\eta(\mathbf{p}^\circ)}{\delta}))$ rounds, where $\lambda_i \leq \frac{1}{2E-1}$ for all i, and $\lambda = \min_i \lambda_i$.*

Although prior work has shown that for suitably small choices of λ_i there is a tatonnement-style price update protocol that converges to equilibrium prices, there has been no prior successful effort to devise and analyze a protocol for which convergence is rapid. Theorem 3.1 provides the first polynomial convergence guarantee for any tatonnement-style protocol with independent price updates, even with λ_i at a fixed value [17].

As the following theorem asserts, this bound is tight for the fixed protocol defined by (1).

THEOREM 3.2 (LOWER BOUND). *For all $\frac{1}{3} \geq \alpha > 0, 1 \geq \beta > 0, E \geq 1$, in weak gross substitutes markets with parameters α, β, E, there are initial prices $\mathbf{p}^\circ > 0$ such that for any final prices $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$, the price update procedure takes $\Omega(\frac{1}{\alpha\beta\lambda}(\log \frac{\eta(\mathbf{p}^\circ)}{\delta}))$ rounds in the One-Time Market setting.*

If the buyers are σ-approximate optimizers as allowed by our computational model, the price updater does not learn $z_i = x_i - w_i$, but only the apparent demand y_i. Thus the update rule becomes

$$p_i \leftarrow p_i(1 + \lambda_i \min\{1, \frac{y_i - w_i}{w_i}\}). \tag{3}$$

[16]The combination of the assumptions of weak gross substitutes and the wealth effect prevent $p^*_i = 0$. To see this, start with equilibrium prices and reduce all prices (except money) by factor $f > 1$. The demand for all goods other than money increases by at least f^β, including the goods with price zero. But the price of these goods has not changed, and other prices have only decreased. This contradicts the weak gross substitute property. Accordingly, it seems reasonable to assume that $\mathbf{p}^\circ > 0$.

[17]It might not be necessary that λ_i be as small as stated in the theorem for all i to get convergence. This depends on the individual changes in rates of demand with respect to price.

Now prices converge to an interval $[\mathbf{p}^L, \mathbf{p}^H]$, where $\mathbf{p}^L$ are the equilibrium prices if the estimates y_i are always maximized, and $\mathbf{p}^H$ are the equilibrium prices for minimum estimates. We let $\eta(p_i)$ denote the distance from p_i to the nearer of $\mathbf{p}^L$ and $\mathbf{p}^H$ if it outside the interval, and set it to 0 if p_i lies within the interval.

THEOREM 3.3 (APPROXIMATE OPTIMIZERS). *In the One-Time Market, the price update protocol given by (3), in weak gross substitutes markets with parameters α, β, E, initial prices $\mathbf{p}^\circ > 0$, and σ-approximate optimizers, for $\sigma < (1-\alpha)^{-\beta} - 1$, yields price vector $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$ in $O(\frac{1}{[1-(1-\alpha)(1+\sigma)^{1/\beta}]\beta\lambda}(\log \frac{\eta(\mathbf{p}^\circ)}{\delta}))$ rounds, where $\lambda_i \leq \frac{1}{2E-1}$ for all i, and $\lambda = \min_i \lambda_i$.*

If the market is liable to change, it is helpful to have a more flexible update protocol. We consider the following. To start, $\lambda_i = \frac{1}{2}$. Let n_i be the number of updates to p_i.

$$p_i \leftarrow p_i + \frac{1}{2^{\lceil \log_4 n_i \rceil}} p_i \min\{1, \frac{z_i}{w_i}\} \qquad (4)$$

THEOREM 3.4. *[Oblivious Market] In the One-Time oblivious market, the price update protocol (4) with b-bounded asynchrony, in weak gross substitutes markets with parameters α, β, E, and initial prices $\mathbf{p}^\circ > 0$, yields price vector $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$ after $O(\frac{b}{\alpha\beta}(E + \log \frac{\max\{1, \eta(p^\circ)\}}{\delta})2)$ rounds.*

It is possible to consider the protocols for the One-Time Markets as algorithms that could be used to compute equilibria. In this case, our analysis simplifies, as we can assume synchrony. Thus, the notion of rounds is not needed and the complexity revolves around iterations in which the price of each good is updated exactly once. The parameter b drops out. It is interesting to note that viewed in this context the number of iterations is independent of both the number of goods and the number of agents. This contrasts with prior work on tatonnement algorithms (e.g. [5]). Admittedly, our analysis does involve other parameters that are absent from these prior analyses.

Ongoing Market Result To state our final result, it is helpful to extend the definition of η to the vector $\mathbf{s}$ of warehouse contents, with the distance to $\mathbf{s}^{\mathbf{F}}$ being measured by η. We first refine the update rule given in (2) to allow p_i to be updated several times during the day, and to allow use of stale information about the demands and the warehouse stock levels. Recall that $\widetilde{x}_i = w_i + \kappa_i h_i$. Let t_a, t_b be times since the last update to p_i. Then

$$p_i' \leftarrow p_i(1 + \lambda_i \min\{1, \frac{x_i(t_a) - \widetilde{x}_i(t_b)}{w_i}\}) \qquad (5)$$

THEOREM 3.5. *Given initial prices $\boldsymbol{p}^o$ and initial warehouse contents $\boldsymbol{s}^o$, the price update (5) results in a price that satisfies $\eta(\boldsymbol{p}) \leq \delta$ and warehouse contents satisfing $\eta(\boldsymbol{s}) \leq \delta$, in*

$$O\left(\frac{1}{\beta\lambda}[\log \eta(\boldsymbol{p}^o) + \log E] + \left(\frac{1}{\kappa} + \frac{1}{\beta\lambda}\right)\left(\log \frac{1}{\beta} + \log \frac{1}{\delta}\right)\right)$$

days, if $\kappa = O(\min_j\{\left(\frac{\beta}{E}\max_j \frac{w_j}{c_j}\right), \frac{\lambda\beta^2}{E}\})$ and $\lambda = O(\frac{1}{E})$.

4. PREVIOUS WORK

To the best of our knowledge, asynchronous price update algorithms have not been considered previously. Further, there has been no complexity analysis of even synchronous tatonnement algorithms with this type of limited information. While Uzawa [27] gave a synchronous algorithm of this type, he only showed convergence, and did not address speed of convergence.

The existence of market equilibria has been a central topic of economics since the problem was formulated by Walras in 1874 [30]. Tatonnement was described more precisely as a differential equation by Samuelson [23]:

$$dp_i/dt = \mu_i z_i. \qquad (6)$$

The μ_i are arbitrary positive constants that represent rates of adjustment for the different prices; they need not all be the same. Arrow, Block, and Hurwitz, and Nikaido and Uzawa [1, 2, 21] showed that for markets of gross substitutes the above differential equation will converge to an equilibrium price.

Unfortunately, for general utility functions (i.e. that do not lead to gross substitutability), the equilibrium need not be stable and the differential equation (and thus also discretized versions) need not converge [24]. Partly in response, Smale described a convergent procedure that uses the derivative matrix of excess demands with respect to prices [26]. Following this, Saari and Simon [22] showed that any price update algorithm which uses an update that is a fixed function of excesses and their derivatives with respect to prices needs to use essentially all the derivatives in order to converge in all markets. However, this is viewed as being an excessive amount of information, in general.

There are really two questions here. The first is how to find an equilibrium, and the second is how does the market find an equilibrium. The first question is partially addressed by the work of Arrow et al. and Smale, and addressed further in papers in operations research (notably Scarf [25] gives a (non-polynomial) algorithm for computing equilibrium prices), and theoretical computer science, where there are a series of very nice results demonstrating equilibria as the solutions to convex programs, or describing combinatorial algorithms to compute such equilibria exactly or approximately. (An early example of a polynomial algorithm for computing market equilibria for restricted settings is [9]. An extensive list of references is given in the surveys [6, 29].)

We are interested in the second question. The differential equations provide a start here, but they ignore the discrete nature of markets: prices typically change in discrete increments, not continuously. In 1960, Uzawa showed that there is a choice of λ for which an obvious discrete analog of (6) does converge [27]. However, determining the right λ depends on knowing properties of the matrix of derivatives of demand with respect to price, or in other words, this requires global information.

More recently, three separate groups have proposed three distinct discrete update algorithms for finding equilibrium prices and showed that their algorithms converge in markets of gross substitutes [18, 8, 5]. However, all of these algorithms use global information. With the exception of [5], none of this work gives (good) bounds on the rate of convergence. The algorithm in Codenotti et al. [5] describes a tatonnement algorithm (albeit not asynchronous); however, it begins by modifying the market by introducing a fictitious player with some convenient properties that capture global information about the market and have a profound effect on market behavior. Even in this transformed setting, the price update step uses a global parameter based on the desired approximation guarantee, and starts with an initial price point that is restricted to lie within a bounded region containing the equilibrium point. Translating their algorithm back into the real market, one can see that it does not meet our definition of simplicity or locality.

There are some auction-style algorithms for finding approximate equilibria which also have a distributed flavor but depend on buyer utilities being separable over the set of goods [12, 13].[18] However,

[18]These algorithms start their computation at a non-arbitrary set of artificially low prices; global information is used for price initial-

these algorithms are not seeking to explain market behavior and not surprisingly do not obey natural properties of markets.

The design and analysis of procedures and convergence to equilibria has been a recent topic of study for game theoretic problems as well. Examples include studying convergence in some network routing and network design games [4, 11, 14, 20]. In partial contrast, it is known that finding equilibria via local search (e.g., via best response dynamics) is PLS-complete in many contexts [17, 10]. Recently, Hart and Mansour [16] gave communication complexity lower bounds to show that in general games, players with limited information require an exponential (in the number of players) number of steps to reach an equilibrium.

The design and analysis of convergent asynchronous distributed protocols has also arisen in network routing, for example [15], and in high latency parallel computing [3], and these lines of work are perhaps the most similar to ours.

5. REMARKS AND FURTHER QUESTIONS

1. In the Ongoing Market, buyers are myopic optimizers; and no assumption is made that sellers are strategic.
2. Discreteness: It would be interesting to extend this work to markets with indivisible goods and discrete prices, for this seems more realistic. (See [7] for some hardness results.)
3. Can this work be extended to incorporate producers?
4. Can the analyses be extended to classes of markets not obeying weak gross substitutes? While the current analysis depends on this assumption, it is not clear that it is necessary. But also note that in markets with linear utilities, for example, our protocol will not converge to equilibrium prices.
5. Devise rapidly convergent protocols that allow the λ_i to increase as well as decrease.
6. It seems plausible that our analysis extends to the Ongoing market, in the Arrow-Debreu setting with parameter $\alpha > 0$. There is a modeling issue though: what do warehouse stocks correspond to? In markets for which each good has a market maker, these can be seen as the stocks they hold.
7. In the Arrow-Debreu market without a numeraire we believe that our algorithm converges to the ray of equilibria at the same rate as for the One-Time Fisher market, but it is not clear what is the meaning of this model in the Ongoing setting.

6. PROOFS OF CONVERGENCE

Here we provide *sketches* of the proofs of our upper bound results.

One-Time Market. For simplicity of notation, we assume throughout this section and the remainder of the paper that $w_i = 1$ for all goods.[19] The implications of this is that now the updates have the form

$$p_i \leftarrow p_i(1 + \lambda_i \min\{z_i, 1\}); \tag{7}$$

the excess demand $z_i = x_i - w_i \geq 0 - 1 = -1$ for all goods i, for any set of prices; and $\alpha_i = \sum_l \frac{x^*_{il} w_{\$l}}{v_l(p^*)}$.

ization; and they work with a global approximation measure — each price update uses the goal approximation guarantee in its update.

[19]This is without loss of generality and may be attained by changing the units of good i.

We want to show that the update rule (7) "improves" the worst price by an $O(\alpha\beta\lambda)$ factor in one round. In particular, this means that if z_i is small, then it is roughly proportional to $\frac{p_i^* - p_i}{p_i^*}$. To demonstrate this we bound x_i by a polynomial in $\frac{p_i}{p_i^*}$ which yields an $O(\frac{|p_i^* - p_i|}{p_i^*})$ bound for $|z_i|$ when p_i is close to p_i^*.

LEMMA 6.1. *Let* $\mathbf{p}, \mathbf{q}$ *with* $\mathbf{q} \leq \mathbf{p}$ *be two price vectors. Then*

$$\frac{x_i(\mathbf{q})}{x_i(\mathbf{p})} \leq \left(\frac{p_i}{q_i}\right)^E.$$

PROOF. Using the definition of E, we have that $\frac{\partial}{\partial p_i}(p_i^E x_i) = E p_i^{E-1} x_i + p_i^E \frac{\partial x_i}{\partial p_i} \geq E p_i^{E-1} x_i - E p_i^{E-1} x_i = 0$. Thus $p_i^E x_i$ is an increasing function of p_i. Consequently, for $p_i' < p_i$ and all other prices fixed, and corresponding demand x_i', $(p_i')^E x_i' \leq p_i^E x_i$ or $\frac{x_i'}{x_i} \leq (\frac{p_i}{p_i'})^E$.

Now, let $\widetilde{\mathbf{q}}$ be the price vector $\mathbf{q}$ with q_i replaced by p_i. By weak gross substitutes, $x_i(\widetilde{\mathbf{q}}) \leq x_i(\mathbf{p})$. Thus $\frac{x_i(\mathbf{q})}{x_i(\mathbf{p})} \leq \frac{x_i(\mathbf{q})}{x_i(\widetilde{\mathbf{q}})} \leq (\frac{p_i}{q_i})^E$. □

LEMMA 6.2. *Suppose that the wealth* v_l *of buyer* l *is multiplied by* $a \geq 1$ *with no change in prices (e.g., by increasing* w_{il} *uniformly for all* i*). Let* $\mathbf{x}'$ *denote the new demand and* $\mathbf{x}$ *the old demand. Then* $\frac{x_{il}'}{x_{il}} \geq a^\beta$.

PROOF. Using the definition of β, we have that $\frac{\partial}{\partial v_l}(v_l^{-\beta} x_{il}) = -\beta v_l^{-\beta-1} x_{il} + v_l^{-\beta} \frac{\partial x_{il}}{\partial v_l} \geq -\beta v_l^{-\beta-1} x_{il} + \beta v_l^{-\beta-1} x_{il} = 0$. Thus $v_l^{-\beta} x_{il}$ is an increasing function of v_l. Consequently, $(a v_l)^{-\beta} x_{il}' \geq v_l^{-\beta} x_{il}$ or $\frac{x_{il}'}{x_{il}} \geq a^\beta$. □

The important component of our analysis is to consider the ratio of p_i to p_i^* and show that over all i the extremes of these ratios get closer to 1 as the protocol proceeds. With this in mind, and without loss of generality, let $1 = \text{argmin}_i \frac{p_i}{p_i^*}$ and $n = \text{argmax}_i \frac{p_i}{p_i^*}$. Also, let $r = p_1^*/p_1$ and $r_i = (p_1^*/p_1)/(p_i^*/p_i)$. Note that $r_i \geq 1$.

LEMMA 6.3. *If* $r \geq 1$, $x_i \geq r^\beta / r_i^E$.

Instead of proving this directly, we prove a more general lemma that will imply this and be useful later.

LEMMA 6.4. *Let* $\mathbf{p}$ *and* $\mathbf{p}'$ *satisfy* $p_i = a b_i p_i'$ *for all* $i \in G$*; and let* x' *and* x *be the corresponding demands.*
(i) If $a \leq 1$ *and* $b_i \geq 1, \forall i$*, then* $x_i \geq a^{-\beta} b_i^{-E} x_i'$.
(ii) If $a \geq 1$ *and* $b_i \leq 1, \forall i$*, then* $x_i \leq a^{-\beta} b_i^{-E} x_i'$.

PROOF. We argue (i). (ii) is symmetric. We change p' to p in several steps, and track the change in demand. At prices $p'a$, the wealth of each buyer is (relatively) increased by $\frac{1}{a}$. Thus by Lemma 6.2, the new demand is at least $x' a^{-\beta}$. Then, increasing price of i by b_i, reduces demand for i by at most b_i^{-E}, by Lemma 6.1. By weak gross substitutes, increasing the other prices to their actual values only increases x_i. □

The proof of Lemma 6.3 follows from Lemma 6.4(i) by choosing $\mathbf{p}' = \mathbf{p}^*$, $a = \frac{1}{r}$ and $b = r_i$.

LEMMA 6.5. *Let* $\lambda_i \leq 1/(2E-1)$ *for all* i*, and let* $\lambda = min_i \lambda_i$*. In Fisher markets, after one update to* p_i *(yielding* p_i'*),*
(i) if $p_1 \geq p_1^*$ *then* $p_i' \geq p_i^*$*; and if* $p_1 < p_1^*$ *then* $p_i'/p_i^* \geq p_1/p_1^*[1 + \lambda \, \min\{1, (\frac{p_1^*}{p_1})^\beta - 1\}]$*;*
(ii) if $p_n \leq p_n^*$ *then* $p_i' \leq p_i^*$*; and if* $p_n > p_n^*$ *then* $p_i'/p_i^* \leq p_n/p_n^*[1 + \lambda \, (\frac{p_n^*}{p_n})^\beta - 1]$.

PROOF. If $r \geq 1$ by Lemma 6.3, $x_1 \geq r^\beta$ and $x_i \geq r^\beta / r_i^E$. Then, it suffices to show that

$$\frac{p_i'}{p_i^*} = \frac{p_i}{p_i^*}[1 + \lambda(x_i - 1)] \geq \frac{p_1}{p_1^*}[1 + \lambda \min\{1, r^\beta - 1\}].$$

Let $s = r^\beta$. It then suffices that

$$f(s, r_i) = r_i[1 + \lambda(s/r_i^E - 1)] - [1 + \lambda \min\{1, s - 1\}] \geq 0.$$

Now $\frac{df}{ds} \leq 0$ for $s \leq 2$ so f is minimized at $s = 2$. $f(2, r_i) = r_i[1+2\lambda/r_i^E - \lambda)] - (1+\lambda)$. $\frac{df}{dr_i} = 1+2\lambda/r_i^E - \lambda - 2E\lambda/r_i^E \geq 0$ if $(2E - \lambda) \leq 1$. Then f is minimized at $r_i = 1$ with $f(2, 1) = 0$ as desired.

For $r < 1$, it suffices that $\frac{p_i}{p_i^*}[1 + \lambda(x_i - 1)] \geq 1$, or that $(r_i/r)[1 + \lambda(r/r_i)^E - 1] \geq 1$, i.e. that $f(1, r_i/r) \geq 0$. But this is already shown, as $r_i/r \geq 1$.

The argument for p_n is similar, but simpler. □

PROOF. (Of Theorem 3.1 for $\alpha = 1$.) We show the rate of convergence of p_1 toward p_1^*. By Lemma 6.2, if $(\frac{p_1^*}{p_1})^\beta \geq 2$, then $\min_i \frac{p_i'}{p_i^*} \geq \frac{p_1}{p_1^*}(1 + \lambda)$. Thus while $(\frac{p_1^*}{p_1})^\beta \geq 2$, in $O(1/\lambda)$ rounds, $\frac{p_i'}{p_i^*}$ doubles. Otherwise, if $p_1 \leq p_1^*$, $\min_i \frac{p_i'}{p_i^*} \geq \frac{p_1}{p_1^*}[1 + \lambda((\frac{p_1}{p_1^*})^\beta - 1)]$. Then

$$\left(\frac{p_1}{p_1^*}\right)^\beta = \left(1 - \frac{p_1^* - p_1}{p_1^*}\right)^{-\beta} \geq 1 + \beta \frac{p_1^* - p_1}{p_1^*}$$

as $p_1^* \geq p_1$. So $\min_i \frac{p_i'}{p_i^*} \geq \frac{p_1}{p_1^*}[1+\lambda\beta(\frac{p_1^* - p_1}{p_1^*})]$. Hence $\min_i \frac{p_i^* - p_i'}{p_i^*} \leq \frac{p_1^* - p_1}{p_1^*}[1+\lambda\beta(\frac{p_1^* - p_1}{p_1^*})]$. Thus, in $O(1/(\lambda\beta))$ rounds, $\min_i \frac{p_i^* - p_i}{p_i^*}$ is reduced by half.

A similar argument applies to the improvement to $\max_i \frac{p_i - p_i^*}{p_i^*}$. □

In the setting of σ-approximate demand, the above arguments are easily modified to show that after one price update if $p_1 \geq p_1^L$, then $p_i' \geq p_i^L$; and if $p_1 < p_1^L$, then $p_i' \geq p_i[1+\lambda \ \min\{1, (\frac{p_1^L}{p_1})^\beta - 1\}]$; while if $p_n \leq p_n^H$, then $p_i' \leq p_i^H$; and if $p_n > p_n^H$, then $p_i' \leq p_i[1 + \lambda\,(\frac{p_n^H}{p_n})^\beta - 1\}]$. Theorem 3.3 for $\alpha = 1$ now follows.

We can show this all works when using stale information, which appears to be the natural approach in an asynchronous setting, Suppose that instead of using the current value of z_i for calculating the update to p_i, the price setter uses a value of z_i from some point since the last update to p_i, or any convex combination of such values. Then, the arguments presented above are readily modified to cover this case, but instead of assuring progress from round to round, they now assure progress every second round.

In the oblivious setting, initially prices may veer off in the wrong direction due to too large λ. Eventually all λ_i become small enough, and then we show that there is a long enough phase of rounds when the smallest λ_i is stable so that sufficient progress is achieved in moving prices towards the equilibrium.

Ongoing Market. For simplicity in the analysis we set all κ_i equal: $\kappa_i = \kappa$.

Our analysis proceeds in two phases. First, the prices enter an interval $[p_i^L/(1+\nu), p_i^H(1+\nu)]$, $1 \leq i \leq n$, where p_i^H are the prices achieving demand $w_i - \kappa s_i^F$, and p_i^L those achieving demand $w_i + \kappa(c_i - s_i^F)$. The analysis for this phase is similar to that used to account for approximate optimizers.

LEMMA 6.6. *Given an initial price vector* $\boldsymbol{p}^o$*, the price vector lies in the range* $[\frac{1}{(1+\nu)}\boldsymbol{p}^L, (1+\nu)\boldsymbol{p}^H]$ *after* $O(\frac{1}{\beta\lambda}\log\frac{\eta(\boldsymbol{p}^o)}{\nu})$ *days.*

For the second phase, our analysis proceeds in days. The behavior of the current day is expressed in terms of parameters $l, \widetilde{l}, \widetilde{\nu}$. The corresponding parameters for the next day are $l', \widetilde{l}', \widetilde{\nu}'$. We let $\widetilde{\mathbf{p}}$ denote the price vector attaining demand $\widetilde{x}$. We prove by induction on the days, that the protocol maintains the following properties:

1. At the start of the day,
$$x_i^* e^{-\widetilde{l}} \leq \widetilde{x}_i \leq x_i^* e^{\widetilde{l}}. \tag{8}$$
2. Let $\widetilde{x}_i(t)$ denote the value of $\widetilde{x}_i$ at time t. Let $t_1 \leq t_2$ be times in the current day and let $t_2 = t_1 + \tau$, where $0 \leq \tau \leq 1$. Then,
$$\widetilde{x}_i(t_1)e^{-\widetilde{\nu}\tau} \leq \widetilde{x}_i(t_2) \leq \widetilde{x}_i(t_1)e^{\widetilde{\nu}\tau}. \tag{9}$$
3. Let $\hat{t}$ be the time of the last update to p_i in the previous day. Then,
$$\widetilde{p}_i(\hat{t})e^{-l} \leq p_i(\hat{t}) \leq \widetilde{p}_i(\hat{t})e^{l}. \tag{10}$$

Let $\rho = \widetilde{\nu}/\beta(1-\mu)$, $\rho_1 = 2E(l+\rho)$, $\rho_2 = \frac{\rho_1 + \widetilde{\nu}}{\widetilde{l}}$, and $\rho_3 = \frac{\beta}{e} - \frac{2\rho}{\lambda l}$. We will show that we can set $l' = l(1-\mu)$, $\widetilde{l}' = \widetilde{l}(1-\mu)$, and $\widetilde{\nu}' = \widetilde{\nu}(1-\mu)$, where $\mu = \min\{\lambda\rho_3, \kappa\frac{1-\rho_2}{e}\}$ (see Lemmas 6.13 and 6.14).

LEMMA 6.7. *Let* $t_1 \leq t_2$ *be times in the current day and let* $t_2 = t_1 + \tau$*, where* $0 \leq \tau \leq 1$*. Then,*
$$\widetilde{p}_i(t_1)e^{-\widetilde{\nu}\tau/\beta} \leq \widetilde{p}_i(t_2) \leq \widetilde{p}_i(t_1)e^{\widetilde{\nu}\tau/\beta}. \tag{11}$$

PROOF. We show the first inequality. By Lemma 6.2, (9), and weak gross substitutes,

$$\left[\max_j \frac{\widetilde{p}_j(t_1)}{\widetilde{p}_j(t_2)}\right]^\beta \leq \frac{\widetilde{x}_i(t_2)}{\widetilde{x}_i(t_1)} \leq e^{\widetilde{\nu}\tau}.$$

□

LEMMA 6.8. *Let t be a time in the current round.*

$$e^{-\rho_1} \leq \frac{x_i(t)}{\widetilde{x}_i(t)} \leq e^{\rho_1}$$

PROOF. We show the second inequality. Let t' be the start of the current day. By Lemma 6.7 applied separately to each interval $[\hat{t}, t']$ and $[t', t]$, (10), and using the weaker value of $\widetilde{\nu}/(1-\mu)$ for today, we have that $e^{-l-\rho} \leq \frac{p_i}{\widetilde{p}_i} \leq e^{l+\rho}$ as $\tau \leq 1$. Let $e^f = \max_j \frac{p_j}{\widetilde{p}_j}$, $e^g = \max_j \frac{\widetilde{p}_j}{p_j}$ at time t. Note that $f + g \leq \rho_1/E$. Applying Lemma 6.4(ii) to change $\widetilde{\mathbf{p}}$ to $\mathbf{p}$ via $a = e^f$ and $b = e^{-f-g}$ and noting that $\frac{1}{e^{f\beta}} \leq 1$ yields the result. □

COROLLARY 6.9. *Let t be the current time and* τ *be t minus the start of current day.*

$$e^{-(\rho_1 + \widetilde{l} + \widetilde{\nu}\tau)} \leq \frac{x_i(t)}{x_i^*} \leq e^{\rho_1 + \widetilde{l} + \widetilde{\nu}\tau}.$$

PROOF. Use (8) and (9) with Lemma 6.8. □

Let x_i^+ denote the largest value of x_i during the current round and x_i^- the smallest one. Then the warehouse stocks grow by at least $(x_i^* - x_i^+)\tau$ and at most $(x_i^* - x_i^-)\tau$ during a length τ subinterval of the current round. Thus:

LEMMA 6.10. *Let* $\tau = t_2 - t_1$.
(i) $\widetilde{x}_i(t_1) - \widetilde{x}_i(t_2) \leq \kappa(x_i^+ - x_i^*)\tau$.
(ii) $\widetilde{x}_i(t_2) - \widetilde{x}_i(t_1) \leq \kappa(x_i^* - x_i^-)\tau$.

LEMMA 6.11. *Relation (9) holds for $\tau \le 1$ if $\rho_1 + \widetilde{l} + \widetilde{\nu} \le 1$, and $\kappa(\rho_1 + \widetilde{l} + \widetilde{\nu})(1 + \widetilde{\nu}/2) \le \frac{1}{e}\widetilde{\nu}(1 - \frac{1}{2}(\rho_1 + \widetilde{l} + \widetilde{\nu}))$.*

PROOF. Here we show the second bound $\widetilde{x}_i(t_2) \le \widetilde{x}_i(t_1)e^{\widetilde{\nu}\tau}$. Equivalently, $\widetilde{x}_i(t_2) - \widetilde{x}_i(t_1) \le \widetilde{x}_i(t_1)(e^{\widetilde{\nu}\tau} - 1)$.

By Lemma 6.10(ii), it suffices to show that $\kappa(x_i^* - x_i^-)\tau \le \widetilde{x}_i(t_1)\ (e^{\widetilde{\nu}\tau} - 1)$. By Corollary 6.9, $x_i^- \ge x_i^* e^{-\{\rho_1 + \widetilde{l} + \widetilde{\nu}\tau\}}$, so it suffices to show that

$$\kappa x_i^*(1 - e^{-\rho_1 - \widetilde{l} - \widetilde{\nu}})\tau \le \widetilde{x}_i(t_1)(e^{\widetilde{\nu}\tau} - 1).$$

Then, it suffices to show that $\kappa(\rho_1 + \widetilde{l} + \widetilde{\nu})\tau \le e^{-\widetilde{l} - \widetilde{\nu}}\widetilde{\nu}\tau$, as $\rho_1 \le 1$. In turn, it suffices that $\kappa(\rho_1 + \widetilde{l} + \widetilde{\nu}) \le \widetilde{\nu}/e$ as $\widetilde{l} + \widetilde{\nu} \le 1$. But this is true by assumption.

The appeal to Corollary 6.9 might appear to create a circularity in the argument, but in fact there is no problem. Strictly, $\widetilde{x}_i(t_2) - \widetilde{x}_i(t_1) = \kappa \int_{t_1}^{t_2} [x_i^* - x_i(t)]dt$, and then we inductively substitute from Corollary 6.9. □

Simple calculus yields the following technical lemma.

LEMMA 6.12. *Let $0 \le \kappa, x, \eta \le 1$. Then*

1. $e^{x(1-\eta\kappa/e)} \ge e^x - \kappa(e^{x\eta} - 1)$.
2. $e^{-x(1-\eta\kappa/e)} \le e^{-x} + \kappa(1 - e^{-x\eta})$.
3. $e^{x(1+\eta\kappa/e)} \le e^x + \kappa(1 - e^{-x\eta})$.

LEMMA 6.13. *If $\rho_2, \kappa, \widetilde{l} \le 1$, then $\widetilde{l}' \le \widetilde{l}\,[1 - \kappa(1 - \rho_2)/e]$.*

Let t_1 be the time at the start of the current day.

PROOF. Case 1: $\widetilde{x}_i(t_1) \ge x_i^*$. Let $\widetilde{x}_i(t_1) = x_i^* e^{\widetilde{l} - \delta}$. Note that by (8), $\delta \ge 0$. By Lemma 6.8 and (9),

$$x_i^- \ge x_i^* e^{\widetilde{l} - \delta} e^{-\widetilde{\nu} - \rho_1} \ge x_i^* e^{\widetilde{l} - \delta - \rho_2 \widetilde{l}}.$$

By Lemma 6.10(ii), $\widetilde{x}_i(t_1) - \widetilde{x}_i(t_1 + 1) \ge \kappa x_i^*(e^{\widetilde{l}(1-\rho_2) - \delta} - 1)$. So

$$\begin{aligned}
\widetilde{x}_i(t_1 + 1) &\le \widetilde{x}_i(t_1) - \kappa x_i^*(e^{\widetilde{l}(1-\rho_2)-\delta} - 1) \\
&\le x_i^*[e^{\widetilde{l} - \delta} - \kappa(e^{\widetilde{l}(1-\rho_2)-\delta} - 1)] \\
&\le x_i^*[e^{\widetilde{l}} - \kappa(e^{\widetilde{l}(1-\rho_2)} - 1)].
\end{aligned}$$

The last inequality follows as the derivative of the previous right hand side with respect to δ is negative for $\delta \ge 0$; it is:

$$x_i^*[-e^{\widetilde{l} - \delta} + \kappa e^{\widetilde{l}(1-\rho_2)-\delta}] = -x_i^* e^{\widetilde{l} - \delta}[1 - \kappa e^{-\widetilde{l}\rho_2}] < 0.$$

So by Lemma 6.12(1), $\widetilde{x}_i(t_1 + 1) \le e^{\widetilde{l}[1 - \kappa(1-\rho_2)/e]}$.

Case 2: $\widetilde{x}_i(t_1) \le x_i^*$. Details omitted. □

LEMMA 6.14. *$l' \le l(1 - \frac{\lambda\beta}{e} + \frac{2\rho}{l})$ if $\lambda e E \le 1$, $\beta \max\{2\rho, l\} \le \widetilde{l} \le 1$, $\widetilde{l} + \rho_1 \le 1$, and $e^{\rho_1 + \widetilde{l} + \widetilde{\nu}} - e^{\widetilde{l}} \le 1$.*

PROOF. The update to p_i uses the rule $p_i' = p_i(1 + \lambda(\frac{x_i - \widetilde{x}_i}{x_i^*}))$ where $x_i = x_i(t_a)$ and $\widetilde{x}_i = \widetilde{x}_i(t_b)$, with t_a, t_b being times since the last update to p_i. Note that we are assuming $\left|\frac{x_i - \widetilde{x}_i}{x_i^*}\right| \le 1$. By Lemma 6.8, (8) and (9), this amounts to $(e^{\rho_1} - 1)e^{\widetilde{l} + \widetilde{\nu}} \le 1$.

Let t_1 be the time at the start of the current day. Without loss of generality, let $1 = \arg\max_i \frac{p_i}{\widetilde{p}_i(t_a)}$ and let $p_1 = \widetilde{p}_1(t_a)e^{l+\rho-r}$.

Case 1: $p_i(t_a) = \widetilde{p}_i(t_a)e^{l+\rho-r-s_i}$.

Note that this implies that $r + s_i \le 2l + 2\rho$. Applying Lemma 6.4(ii) to change $\widetilde{p}_i(t_a)$ to $p_i(t_a)$ using $a = e^{l+\rho-r}$ and $b = \frac{1}{e^{s_i}}$ yields

$$\frac{x_i(t_a)}{\widetilde{x}_i(t_a)} \le e^{-\beta[l+\rho-r]}e^{Es_i} \le e^{-\beta l - \beta\rho + Es_i + \beta r} \qquad (12)$$

By (9), $\widetilde{x}_i(t_a) \le \widetilde{x}_i(t_1)e^{|t_a - t_1|\beta\rho} \le \widetilde{x}_i(t_1)e^{\beta\rho}$ (note that t_a may occur in the previous round). Similarly, $\widetilde{x}_i(t_a)e^{-\beta\rho} \le \widetilde{x}_i(t_b)$. Let t be the update time. Then

$$\begin{aligned}
p_i'(t) &= p_i(t)\left[1 + \lambda\left(\frac{x_i(t_a) - \widetilde{x}_i(t_b)}{x_i^*}\right)\right] \\
&\le p_i(t)\left(1 + \lambda\frac{\widetilde{x}_i(t_1)}{x_i^*}\frac{\widetilde{x}_i(t_a)}{\widetilde{x}_i(t_1)}\left[\frac{x_i(t_a)}{\widetilde{x}_i(t_a)} - \frac{\widetilde{x}_i(t_b)}{\widetilde{x}_i(t_a)}\right]\right) \\
&\le p_i(t)\left(1 + \lambda\frac{\widetilde{x}_i(t_1)}{x_i^*}\frac{\widetilde{x}_i(t_a)}{\widetilde{x}_i(t_1)}\left[e^{-\beta l + Es_i + \beta r - \beta\rho} - e^{-\beta\rho}\right]\right)
\end{aligned}$$

Case 1.1: $\beta l \ge Es_i + \beta r$.

$$\begin{aligned}
\frac{p_i'(t)}{\widetilde{p}_i(t)} &\le \frac{p_i(t)}{\widetilde{p}_i(t)}\{1 + \lambda\frac{\widetilde{x}_i(t_1)}{x_i^*}\frac{\widetilde{x}_i(t_a)}{\widetilde{x}_i(t_1)}[e^{-\beta l + Es_i + \beta r - \beta\rho} \\
&\qquad - e^{-\beta\rho}]\} \\
&\le e^{l + 2\rho - r - s_i}[1 - \lambda e^{-\widetilde{l}}e^{-2\beta\rho}(1 - e^{-\beta l + Es_i + \beta r})]
\end{aligned}$$

using Case 1 assumption and Lemma 6.7 to bound $p_i(t)/\widetilde{p}_i(t)$, (8) for $\widetilde{x}_i(t_1)/x_i^*$, and (9) for $\widetilde{x}_i(t_a)/\widetilde{x}_i(t_1)$.

We differentiate w.r.t. s_i and then w.r.t. r to show that the right hand side is maximized at $s_i = 0$ and $r = 0$; the derivative w.r.t. s_i is:

$$\begin{aligned}
&-e^{l+2\rho-r-s_i}[1 - \lambda e^{-\widetilde{l}}e^{-2\beta\rho}(1 - e^{-\beta l + Es_i + \beta r}) \\
&\qquad -\lambda E e^{-\widetilde{l}}e^{-2\beta\rho}e^{-\beta l + Es_i + \beta r}] \\
&\le -e^{l+2\rho-r-s_i}[(1-\lambda) - \lambda(E-1)] \\
&\le 0 \quad \text{if } \lambda E \le 1.
\end{aligned}$$

Differentiating with respect to r gives similar results. We obtain

$$\begin{aligned}
\frac{p_i'(t)}{\widetilde{p}_i(t)} &\le e^{l+2\rho}[1 - \lambda e^{-\widetilde{l}}e^{-2\beta\rho}(1 - e^{-\beta l})] \\
&\le e^{l+2\rho-\widetilde{l}-2\beta\rho-\beta l}\left[e^{\widetilde{l}+2\beta\rho+\beta l} - \lambda\left(e^{\beta l} - 1\right)\right] \\
&\le e^{l+2\rho-\widetilde{l}-2\beta\rho-\beta l}e^{[\widetilde{l}+2\beta\rho+\beta l] - \frac{\lambda}{e}\beta l} \\
&\qquad \text{by Lemma 6.12(1), assuming } \widetilde{l} + 2\beta\rho + \beta l \le 1 \\
&\le e^{l+2\rho-\lambda\beta l/e} \\
&\le e^{l(1-\lambda\beta/e+2\rho/l)}.
\end{aligned}$$

Case 1.2: $\beta l \le Es_i + \beta r$. Details omitted.

Case 2: $p_i = \widetilde{p}_i(t_a)e^{-l-\rho+r+s_i}$. Details omitted. □

LEMMA 6.15. *The conditions of Lemmas 6.11, 6.13 and 6.14 hold if $\widetilde{l} = 8El \le \frac{1}{3}$, $\widetilde{\nu} = \frac{\lambda\beta 2(1-\mu)}{4e}l$, $\beta \le 1$, $\lambda \le 1/eE$, $\kappa \le \frac{\lambda\beta 2}{65e^2E}$.*

PROOF. The conditions that need to be met are:

$$\rho_1 + \widetilde{l} + \widetilde{\nu} \le 1 \quad \text{(from Lemma 6.11)} \quad (13)$$

$$\kappa(\rho_1 + \widetilde{l} + \widetilde{\nu})(1 + \frac{\widetilde{\nu}}{2}) \le \widetilde{\nu}(1 - (\rho_1 + \widetilde{l} + \widetilde{\nu})/2)/e \quad \text{(from Lemma 6.11)} \quad (14)$$

$$\rho_2 = \frac{\rho_1 + \widetilde{\nu}}{\widetilde{l}} \le 1 \quad \text{(from Lemma 6.13)} \quad (15)$$

$$e^{\rho_1 + \widetilde{\nu} + \widetilde{l}} - e^{\widetilde{l}} \le 1 \quad \text{(from Lemma 6.14)} \quad (16)$$

The other constraints are subsumed by these. (13)–(16) are readily verified given the constraints on $l, \widetilde{l}, \widetilde{\nu}$, and κ. □

The remaining issue is the requisite bound on κ in order that the first round end with $\widetilde{l}$ and l small enough.

LEMMA 6.16. *If $\kappa \le \frac{\beta}{144E} \min_i \frac{w_i}{c_i}$, then $l \le \frac{1}{24E}$ and $\widetilde{l} \le \frac{1}{3}$ at the start of the second phase, after $O[\frac{1}{\beta\lambda}(\log \eta(\mathbf{p}_0) + \log E)]$ days.*

PROOF. At all times the target demand for the ith good lies in the range $x_i^* \pm \kappa c_i = x_i^*(1 \pm \kappa c_i/w_i)$. Let us rewrite this as $e^{-a} \le \frac{\widetilde{x}_i}{x_i^*} \le e^a$, for all i, for a suitable a. Then, by Lemma 6.2, $e^{-a/\beta} \le \frac{\widetilde{p}_i}{p_i^*} \le e^{a/\beta}$, for all i. Since in the first phase p_i approaches the interval $[\widetilde{p}_i^L, \widetilde{p}_i^H] \subseteq [e^{-a/\beta}\widetilde{p}_i, e^{a/\beta}\widetilde{p}_i]$ we readily obtain that at the end of Phase 1, $e^{-3a/\beta} \le \frac{p_i}{\widetilde{p}_i} \le e^{3a/\beta}$ for all i (set $1 + \nu = e^{a/\beta}$ in Lemma 6.6).

For (10), we need $3a/\beta \le \frac{1}{24E}$. Hence $\nu = O(1/E)$, which yields the stated running time by Lemma 6.6.

Finally, for all i, as $a \le 1$, it suffices to have $\kappa c_i/w_i \le a/2$, which gives the stated bound on κ. □

PROOF. (Of Theorem 3.5.) First, we note that $\rho_3 = \frac{\beta}{2e}$ and $\rho_2 \le \frac{1}{2}$. Consequently, the improvement to $l, \widetilde{\nu}$ and $\widetilde{l}$ in Phase 2 is by a factor of at least $1 - \mu = \min\{\theta(\lambda\beta), \theta(\kappa)\}$ per round. Thus, in $O(\max \frac{1}{\kappa}, \frac{1}{\lambda\beta})$ rounds the upper bounds of $e^{l+\rho} - 1$ on $\max_i \frac{\widetilde{p}_i - p_i}{p_i}$ and on $\max_i \frac{p_i - \widetilde{p}_i}{p_i}$ halve, as do the upper bounds of $e^{\widetilde{l}+\widetilde{\nu}}$ on $\max_i \frac{x_i^* - \widetilde{x}_i}{\widetilde{x}_i}$ and $\max_i \frac{\widetilde{x}_i - x_i^*}{x_i^*}$.

By Lemma 6.2,

$$\begin{aligned}
\max_i \frac{\widetilde{p}_i}{p_i^*} &\le \left(\max_i \frac{x_i^*}{\widetilde{x}_i}\right)^{\frac{1}{\beta}} \le 1 - \left(\frac{x_i^* - \widetilde{x}_i}{x_i^*}\right)^{\frac{1}{\beta}} \\
&\le 1 + \frac{1}{\beta}\left(\frac{x_i^* - \widetilde{x}_i}{\widetilde{x}_i}\right) + \frac{1}{\beta}\left(\frac{x_i^* - \widetilde{x}_i}{\widetilde{x}_i}\right)^{e^{\widetilde{l}+\widetilde{\nu}}} + \cdots \\
&\le 1 + \frac{1}{\beta(1 - e^{\widetilde{l}+\widetilde{\nu}})}\left(\frac{x_i^* - \widetilde{x}_i}{\widetilde{x}_i}\right) \\
&= 1 + O(\frac{1}{\beta})\left(\frac{x_i^* - \widetilde{x}_i}{\widetilde{x}_i}\right).
\end{aligned}$$

A similar bound holds for $\min_i \frac{\widetilde{p}_i}{p_i^*}$. It follows that in at most an additional $O(\log \frac{1}{\beta})$ rounds, the same reduction to $\max_i \frac{p_i - p_i^*}{p_i}$ as to $\max_i \frac{p_i - \widetilde{p}_i}{p_i}$ is achieved and likewise to $\max_i \frac{p_i^* - p_i}{p_i^*}$ compared to $\max_i \frac{\widetilde{p}_i - p_i}{\widetilde{p}_i}$. Thus using $O\left(\left(\frac{1}{\kappa} + \frac{1}{\beta\lambda}\right)\left(\log \frac{1}{\beta} + \log \frac{1}{\delta}\right)\right)$ rounds in the second phase, a price $\eta(\mathbf{p}) \le \delta$ and a warehouse contents $\eta(\mathbf{s}) \le \delta$ are achieved. Lemma 6.16 bounds the cost of Phase 1. □

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