Lecture 14
APPROXIMATE ZEROS

One of the most effective methods to approximate the zeros of a function \( f(z) \) is to apply Newton's iteration. If \( f(z^*) = 0 \), we call \( z \in \mathbb{C} \) an approximate zero associated with \( z^* \) if, applying Newton's iteration to \( z \), we converge to \( z^* \). More precisely, let \( f : \mathbb{C} \to \mathbb{C} \) be an analytic function given by

\[ f(z) = \sum_{i=0}^{\infty} a_i z^i. \]

The Newton operator \( N_f : \mathbb{C} \to \mathbb{C} \) is given by

\[ N_f(z) = \begin{cases} 
  z & \text{if } f'(z) = 0 \\
  z - \frac{f(z)}{f'(z)} & \text{else.}
\end{cases} \quad (1) \]

We call \( z_0 \in \mathbb{C} \) an approximate zero of \( f \) if there is a zero \( z^* \) of \( f \) such that for all \( n \geq 1 \),

\[ |z_n - z^*| < |z_{n-1} - z^*|, \quad (2) \]

where \( z_n = N_f(z_{n-1}) \). Call \( z^* \) the associated zero of \( z_0 \). If, instead of (2), we have

\[ |z_n - z^*| \leq \left( \frac{1}{2} \right)^{2^n-1} |z_0 - z^*|. \quad (3) \]

then \( z_0 \) is a strong approximate zero of \( f \).

Let \( \varrho_0(z^*) \) denote the Newton radius of a root \( z^* \), defined as the smallest positive real number such that every \( z \) which satisfies \( |z - z^*| < \varrho_0(z^*) \) is an approximate zero of \( z^* \). The open ball of radius \( \varrho_0(z^*) \) centered at \( z^* \) is called the Newton basin of \( z^* \), denoted \( NB_0(z^*) \). If we use the strong version of approximate zeros instead, then we obtain strong Newton radius and strong Newton basin, respectively; the corresponding notations are \( \varrho_1(z^*) \) and \( NB_1(z^*) \). These concepts were essentially introduced by Smale.

It is well-known that \( \varrho_0(z^*) \) is well-defined; also \( \varrho_1(z^*) \) is undefined when \( z^* \) is a multiple zero. We are interested in bounds \( B \) such that if \( |z - z^*| < B \) then we conclude that \( z \) is an approximate zero. There are two kinds of such bounds: \( \text{a posteriori} \) bounds depends on \( f, z^* \) and \( z \), i.e., \( B = B(f, z^*, z) \). But \( \text{a priori} \) are \( B = B(f, z^*) \) do not depend on \( z \).
• Smale proved an upper bound of

$$\rho_*(z^*) \leq \frac{(3 - \sqrt{7})/2}{\gamma(f, z^*)}$$

for the strong Newton radius. I want to prove a corresponding upper bound on the Newton radius $$\rho(z^*)$$.

• Is there anything to be gained if I combine Propositions 2 and 3?

• Proposition 1 – what is the correct general formulation if I want a absolute bits?

• What is a practical way to estimate $$\gamma(f, z^*)$$? If you are at a value $$z$$, how do you know you are an approximate zero?

• Can we improve Smale’s bound if we assume $$f$$ is a real polynomial? Basically, in our arguments, instead of $$|z|^2$$ we can use $$z^2$$.

• This was originally implemented in Core’s Newton refine: A Smale bound which is an aposteriori condition. Applying Newton iteration to any point $$z$$ satisfying this condition we are sure to converge to the nearest root in a certain interval of $$z$$. The condition is that, for all $$k > 1$$,

$$\frac{|p^k(z)|^{1/(k-1)}}{|k! p'(z)|} < \frac{1}{8} \cdot \frac{|p'(z)|}{|p(z)|}.$$ 

```cpp
bool smaleBound(const Polynomial<NT> * p, BigFloat z)
{
    int deg = p[0].getTrueDegree();
    BigFloat max, temp, temp1, temp2;
    // Polynomial<NT> q;
    temp2 = p[1].eval(z);
    temp = core_abs(temp2/p[0].eval(z))/8;
    max = 0;
    for(int k = 2; k <= deg; k++){
        // q.differentiate();
        temp1 = core_abs(p[k].eval(z)/(factorial(k)*temp2));
        if(k-1 == 2)
            temp1 = sqrt(temp1);
        else
            temp1 = root(temp1, k-1);
        max = core_max(max, temp1);//root(temp1, k-1));
        if(max > temp) return false; // This makes things faster
    //in the starting stages
    }
    return (max < temp);
}
```

• The following is claimed to be better than the previous criteria, and is the current implementation in Core Library for Newton refine. It is claimed to be more computable than the previous criteria. BUT how do we know that this is better than yap’s bound? The criterion is

$$\|f\|_{\infty} \frac{f(z)}{f'(z)^2} \frac{\phi'(|z|)^2}{\phi(|z|)}$$
where $\phi(r) = \sum_{i=0}^{m} r^i$, and $m = \deg(f)$. This is given as Theorem B in Smale86. For our implementation we calculated an upper bound for $\phi(r)$, $r > 0$,

$$\phi(r) < \frac{(m \cdot r^{m+1} + 1)^2}{(r-1)^3 \cdot (r^{m+1} - 1)}$$


```c
bool smaleBound(BigFloat z){
    int m = seq[0].getTrueDegree();
    BigFloat temp = seq[0].eval(z)/power(seq[1].eval(z),2);
    temp = core_abs(temp)*seq[0].height();
    //temp = $||f||_{\infty} \frac{f(z)}{f'(z)^2}$
    BigFloat x = core_abs(z);
    BigFloat temp1 = power(x, m+1);//$x^{m+1}$
    BigFloat temp2 = power(m*temp1 + 1, 2);//$(m \cdot x^{m+1} + 1)^2$
    temp2 = temp2/(power(x - 1, 3)*(temp1 - 1));//$(m \cdot x^{m+1} + 1)^2$
    //-----------------------
    // $(x-1)^3 \cdot (x^{m+1} - 1)$
    if(8*temp * temp2 < 1)
        return true;
    else
        return false;
}
```

- Consider Newton’s iteration on $f(x)$. Assume $x_0$ is in the strong Newton basin of $x^*$. Suppose $x_1 = x_0 + \delta$ where $\delta = -f(x_0)/f'(x_0)$. Prove that $|x_1 - x^*| < |\delta|$.

§2. Á Posteriori Condition for Approximate Zero

Suppose $z_0 \in NB_1(z^*)$. Then to obtain an absolute precision of $a$ bits it is sufficient to do

$$\lg(1 + \max\{0, a - \lg|z_0 - z^*|\})$$

Newton steps starting from $z_0$.

But how can we know that $z_0 \in NB_1(z^*)$? Typically, by applications of Sturm sequence theory, we can upper bound $|z_0 - z^*|$. So we would like to know how small should $|z_0 - z^*|$ be. In this section, we shall prove an á posteriori bound due to Smale. Recall that $z^*$ is a critical point of $f$ if $f'(z^*) = 0$. We assume that $z^*$ is a non-critical point in this section. For any non-critical value $z \in \mathbb{C}$, define

$$\gamma(f, z) := \sup_{k>1} \left| \frac{f^{(k)}(z)}{f'(z)k!}\right|^{1/(k-1)} \tag{4}$$

Our main result is the following:

**Theorem 1** If $f$ is an analytic function from $\mathbb{C} \rightarrow \mathbb{C}$ and $z^*$ any root of $f$ such that $f'(z^*) \neq 0$ then $z_0 \in \mathbb{C}$ is a strong approximate zero of $f$ with associated zero $z^*$ if

$$|z_0 - z^*| \leq \frac{3 - \sqrt{7}}{2\gamma(f, z^*)} \leq 0.17712 \frac{1}{\gamma(f, z^*)}.$$
To use this theorem, we need upper bounds on $\gamma(f, z^*)$. The proof of this theorem is developed via the following lemmas. First, let 

$$u(z, z^*) := \gamma(f, z^*)|z - z^*|. \tag{5}$$

**Lemma 2** If $z \in \mathbb{C}$ is such that $u(z, z^*) < 1$ then 

$$|N_f(z) - z^*| \leq \left| \frac{f'(z^*)}{f'(z)} \frac{u(z, z^*)}{(1 - u(z, z^*))^2} |z - z^*| \right|$$

**Proof.** Using Taylor’s expansion on $f(z)$ and $f'(z)$ around $z^*$ we get 

$$f(z) = f'(z^*)(z - z^*) + \frac{f''(z^*)}{2}(z - z^*)^2 + \ldots = \sum_{k=1}^{\infty} \frac{f^{(k)}(z^*)}{k!}(z - z^*)^k \tag{6}$$

$$f'(z) = f'(z^*) + f''(z^*)(z - z^*) + \frac{f'''(z^*)}{2}(z - z^*)^2 + \ldots = \sum_{k=1}^{\infty} \frac{f^{(k)}(z^*)}{(k - 1)!}(z - z^*)^{k-1} \tag{7}$$

$$|N_f(z) - z^*| = |z - z^* - f(z)/f'(z)|$$

$$= \left| \frac{1}{f'(z)} \left| f'(z) (z - z^*) - f(z) \right| \right|$$

$$= \left| \frac{1}{f'(z)} \left| \sum_{k=1}^{\infty} \frac{f^{(k)}(z^*)}{(k - 1)!}(z - z^*)^k - \sum_{k=1}^{\infty} \frac{f^{(k)}(z^*)}{k!}(z - z^*)^k \right| \right|$$

$$= \left| \frac{1}{f'(z)} \left| \sum_{k=1}^{\infty} \frac{f^{(k)}(z^*)}{k!}(z - z^*)^k(k - 1) \right| \right|$$

$$\leq \left| \frac{f'(z^*)}{f'(z)} \right| \left| \sum_{k=1}^{\infty} \frac{f^{(k)}(z^*)}{f'(z^*)k!}(z - z^*)^{k-1}(k - 1) \right|$$

$$\leq \left| \frac{f'(z^*)}{f'(z)} \right| \left| \sum_{k=1}^{\infty} (k - 1) \gamma(z, z^*)^{k-1} |z - z^*|^{k-1} \right|$$

$$= \left| \frac{f'(z^*)}{f'(z)} \right| \left| \sum_{k=1}^{\infty} (k - 1) u(z, z^*)^{k-1} \right|$$

$$= \left| \frac{f'(z^*)}{f'(z)} \right| \left| \frac{u(z, z^*)}{(1 - u(z, z^*))^2} |z - z^*| \right|$$

where the last equality follows from $u(z, z^*) < 1$ and the fact 

$$\frac{z}{(1 - z)^2} = \sum_{i=1}^{\infty} iz^i. \tag{8}$$

This can be seen by differentiating the geometric series $\frac{1}{1-z} = \sum_{i=1}^{\infty} z^i$ to obtain 

$$\frac{1}{(1-z)^2} = \sum_{i=1}^{\infty} iz^{i-1}. \quad \text{Q.E.D.}$$

Next, we get an upperbound on $|\frac{f'(z^*)}{f'(z)}|$.

**Lemma 3** Suppose $z$ is such that $u(z, z^*) < 1 - 1/\sqrt{2} = 0.29289 \ldots$, hence less than one, then 

$$\left| \frac{f'(z^*)}{f'(z)} \right| < \frac{(1 - u(z, z^*))^2}{\phi(z, z^*)}$$

where $\phi(z, z^*) := 1 - 4u(z, z^*) + 2u(z, z^*)^2$. 

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Proof.

\[
\left| \frac{f'(z)}{f'(z)} \right| = \left| \frac{f'(z^*)}{f'(z^*) + \sum_{k=2}^{\infty} \frac{f^{(k)}(z^*)(z-z^*)^{k-1}}{(k-1)!}} \right|
\]

\[
= \left| \frac{1}{1 + \sum_{k=2}^{\infty} \frac{f^{(k)}(z^*)(z-z^*)^{k-1}}{f'(z^*)(k-1)!}} \right|
\]

\[
= \left| \frac{1}{1-B} \right|
\]

\[
= \sum_{i=0}^{\infty} B^i
\]

\[
\leq \sum_{i=0}^{\infty} |B|^i
\]

where

\[
|B| = \sum_{k=2}^{\infty} \left| \frac{f^{(k)}(z^*)(z-z^*)^{k-1}}{f'(z^*)(k-1)!} \right| \quad (9)
\]

\[
\leq \sum_{k=2}^{\infty} \left| \frac{f^{(k)}(z^*)(z-z^*)^{k-1}}{f'(z^*)(k-1)!} \right| \quad (10)
\]

\[
\leq \sum_{k=2}^{\infty} k(\gamma(f, z^*)|z-z^*|)^{k-1} \quad (11)
\]

\[
= \sum_{k=2}^{\infty} ku(z, z^*)^{k-1} \quad (12)
\]

\[
= \frac{1}{(1-u(z,z^*))^2 - 1}, \quad (13)
\]

again using \(u(z, z^*) < 1\) and (8).

By assumption \(u(z, z^*) < 1 - 1/\sqrt{2}\) which implies \(\frac{1}{(1-u(z,z^*))^2} - 1 < 1\). Thus we get the following bound

\[
\sum_{i=0}^{\infty} |B|^i < \sum_{i=0}^{\infty} \left( \frac{1}{(1-u(z,z^*))^2} - 1 \right)^i
\]

\[
= \frac{1}{2 - \frac{1}{(1-u(z,z^*))^2}}
\]

\[
= \frac{1 - u(z,z^*)^2}{\phi(z,z^*)}
\]

Q.E.D.

Substituting the result of Lemma 3 in Lemma 2 we get

**Lemma 4** If \(z \in \mathbb{C}\) is such that \(u(z, z^*) < 1 - 1/\sqrt{2}\) then

\[
|N_f(z) - z^*| \leq \frac{u(z, z^*)}{\phi(z,z^*)} |z - z^*| = \frac{\gamma(f,z^*)}{\phi(z,z^*)} |z - z^*|^2 \quad (14)
\]
We will use induction to prove the following

\[ |N_f^{(k)}(z) - z^*| \leq \left( \frac{u(z, z^*)}{\phi(z, z^*)} \right)^{2^{k-1}} |z - z^*|. \] (15)

The basis \((k = 1)\) is shown by Lemma 4. To carry out the induction, we need the next lemma.

**Lemma 5** If \(z \in \mathbb{C}\) is such that \(u(z, z^*) < \frac{\sqrt{5} - \sqrt{17}}{4} \approx 0.21922\ldots\), then

\[ u(N_f(z), z^*) < u(z, z^*) \] (16)

and

\[ \phi(N_f(z), z^*) > \phi(z, z^*) \] (17)

**Proof.** We note that \(\frac{\sqrt{5} - \sqrt{17}}{4} < 1 - 1/\sqrt{2}\), hence \(z\) satisfies the criterion for Lemma 4

\[ u(N_f(z), z^*) = \gamma(f, z^*)|N_f(z) - z^*| \]
\[ \leq \gamma(f, z^*) \frac{u(z, z^*)}{\phi(z, z^*)} |z - z^*| \]
\[ = u(z, z^*) \frac{u(z, z^*)}{\phi(z, z^*)}. \]

Thus we desire \(\frac{u(z, z^*)}{\phi(z, z^*)} < 1\), but this is satisfied since \(\frac{\sqrt{5} - \sqrt{17}}{4}\) is the smallest positive root of the quadratic equation \(\phi(z, z^*) - u(z, z^*) > 0\) with \(u(z, z^*)\) being the variable. Equation (17) holds because \(\phi(z, z^*)\) is a decreasing function of \(u(z, z^*)\) in the range \(0 \leq u(z, z^*) \leq \frac{\sqrt{5} - \sqrt{17}}{4}\). Q.E.D.

We continue in our proof of Equation (15): suppose the hypotheses holds for \(k - 1\), i.e,

\[ |N_f^{(k-1)}(z) - z^*| \leq \left( \frac{u(z, z^*)}{\phi(z, z^*)} \right)^{2^{k-1}} |z - z^*| \] (18)

From Lemma 4 and Lemma 5 we get that

\[ |N_f(N_f^{(k-1)}(z)) - z^*| \leq \frac{\gamma(f, z^*)}{\phi(N_f(z), z^*)} |N_f^{(k-1)}(z) - z^*|^2, \quad \text{(induction hypotheses, (18))} \]
\[ \leq \left( \frac{u(z, z^*)}{\phi(z, z^*)} \right)^{2^{k-1}} |z - z^*| \]

Equation (15) and the criterion \(u(z, z^*) < \frac{\sqrt{5} - \sqrt{17}}{4}\) ensures the convergence of the sequence \((z_k)\). However, to guarantee that \(z_0\) is a strong approximate zero we need the more stronger criterion

\[ \frac{u(z, z^*)}{\phi(z, z^*)} < \frac{1}{2} \]
\[ \Leftrightarrow 2u(z, z^*)^2 - 6u(z, z^*) + 1 > 0 \]

Since \(\frac{3 - \sqrt{7}}{2}\) is the first root of the quadratic equation above, it is sufficient that

\[ u(z, z^*) < \frac{3 - \sqrt{7}}{2}. \]
Since $\frac{3-\sqrt{7}}{2} < \frac{5-\sqrt{17}}{4}$, this means Lemmas 2, 3, 4 and 5 hold for $z_0$. This concludes the proof of Theorem 1.

Exercises

Exercise 2.1: Prove the following Newton iteration on $f(x)$. Assume $x_0$ is in the strong Newton basin of $x^\ast$. Suppose $x_1 = x_0 + \delta$ where $\delta = -f(x_0)/f'(x_0)$. Prove that the interval $(x_1 - |\delta|, x_1 + |\delta|)$ contains $x^\ast$.

Exercise 2.2:

End Exercises

§3. Á Priori Strong Approximate Zeros

We shall Smale’s Theorem: If $f(x) \in \mathbb{Z}[x]$ is square-free of degree $m$ with real zero $x^\ast$, and for any $x_0$ with

$$|x_0 - x^\ast| \leq \frac{3 - \sqrt{7}}{2\gamma(f,x^\ast)}$$

where

$$\gamma(f,x) := \sup_{k > 1} \left| \frac{f^{(k)}(x)}{f'(x)k!} \right|^{1/(k-1)}$$

then $x_0$ is a strong approximate zero of $f$ with associated zero $x^\ast$.

We use the lower bound for

$$|f'(x^\ast)| \geq \frac{1}{m^{m-3/2} \|f\|_{\infty}^{m-2}}$$

and upper bound (from Vikram)

$$|f^{(k)}(x^\ast)| \leq \frac{m!}{(m-k)!} M^{m+1}$$

where $M = 1 + \|f\|_{\infty}$, we get

$$\gamma(f,x^\ast) \leq \sup_{k > 1} \left( \frac{m!}{k!(m-k)!} M^{m+1} m^{m-(3/2)} \|f\|_{\infty}^{m-2} \right)^{1/(k-1)}$$

$$< \sup_{k > 1} \left( \left( \frac{m}{k} \right) M^{2m-1} m^{m-(3/2)} \right)^{1/(k-1)}$$

$$\leq M^{2m-1} m^{m-(3/2)} \sup_{k > 1} \left( \frac{m}{k} \right)^{1/(k-1)}$$

$$< \frac{1}{2} M^{2m-1} m^{m+(1/2)}.$$
Define the \textbf{Newton radius} \( \rho(x^*) \) of a zero \( x^* \) of a polynomial \( f(x) \) to be the largest \( r > 0 \) such that for all \( x \), if \( |x - x^*| < r \) then the Newton iteration starting from \( x \) will converge to \( x^* \). The \textbf{Newton basin} of \( x^* \) comprise all those points whose distance from \( x^* \) is less than \( \rho(x^*) \).

Thus we have shown:

\begin{theorem}
Let \( f \in \mathbb{Z}[x] \) be a square free polynomial of degree \( m \) and \( M = \| f \|_\infty + 1 \). Every zero \( x^* \) of \( f \) has Newton radius at least
\[
\delta(f) := \frac{3 - \sqrt{7}}{m^{m+(1/2)}M^{2m-1}}.
\]
\end{theorem}

\textit{Proof.} We just apply the bound just shown above,
\[
\gamma(f, x^*) \leq \frac{1}{2} M^{2m-1} m^{m+1/2},
\]
to Smale’s requirement for a strong approximate zero of \( x^* \). \hfill \textbf{Q.E.D.}

Since Newton basins of distinct zeros are disjoint, we conclude:

\begin{corollary}
The distance between any two distinct zeros of \( f \) is at least
\[
2\delta(f) := \frac{2(3 - \sqrt{7})}{m^{m+(1/2)}M^{2m-1}}.
\]
\end{corollary}

\section{Path Lifting Method for Zeros}

We follow Kim and Sutherland (1991) in their exposition of the path-lifting method for approximating zeros of a polynomial. The idea is to view the polynomial \( f \) as a map \( f : \mathbb{C}_s \to \mathbb{C}_t \) where \( \mathbb{C}_s, \mathbb{C}_t \) are “source” and “target” spaces of the map. Given \( z_0 \in \mathbb{C}_s \), we compute \( w_0 = f(z_0) \in \mathbb{C}_t \). We connect \( w_0 \) to the origin of \( \mathbb{C}_t \) by a straightline path \( \pi \). We then choose the proper branch of \( f^{-1} \) to “lift” this segment back to \( \mathbb{C}_s \). The result is a path \( f^{-1}(\pi) \) that connects \( z_0 \) to a zero \( z^* = f^{-1}(0) \in \mathbb{C}_s \) of \( f \). So we want to take a sequence \( w_n \) of points along the path \( \pi \) and to compute their approximate inverses \( z_n \approx f^{-1}(w_n) \), where
\[
z_{n+1} = z_n - \frac{f(z_n) - w_n}{f'(z_n)}.
\]
This method will converge to the zero \( z^* \) if there is a wedge of ray in \( \mathbb{C}_t \) on which there is a well-defined branch of the inverse \( f_{z^*}^{-1} \).

\begin{figure}
\textbf{FIGURE: path lifting method}
\end{figure}

In Kim and Sutherland (1991), Lemma 1.3 states:

\begin{lemma}
Let
\[
\alpha(f, z) := \max_{k \geq 1} \left| \frac{f(z)}{f'(z)} \right| \left| \frac{f^{k-1}(z)}{k!f'(z)} \right|^{1/(k-1)}.
\]
If \( \alpha(f, z) < 1/8 \), then \( z \) is an approximate zero of \( f \). They define \( z_0 \) to be an approximate zero if for some zero \( z^* \),
\[
|z_n - z^*| < 8(1/2)^2^n |z_0 - z^*|
\]
and \( z_n \) is the \( n \)-th iterate of the Newton iteration at \( z_0 \).
\end{lemma}

They attribute this to Smale86 (Theorem A in the paper in ”Emerging...” book)

\section{Multiple Roots}

So far, we have assumed that \( f'(z^*) \neq 0 \). There are several ways to overcome this assumption. When \( f(z) \) is a polynomial, we may be able to remove multiple roots by using the square-free part of \( f(z) \) instead.
1 Newton for Multiple Roots

Notes from Gilbert, www.math.uwaterloo.ca/wgilbert/Research/NewtonMethods/Sect05.html. He has a paper on this which you can cite.

2 Dynamics of Iteration

Let \( N(z) \) be an iteration operator. The set \( \{ z, N(z), N^2(z), \ldots \} \) is called the orbit of \( z \). We call \( z \) a periodic point of \( N(z) \) if \( N^p(z) = z \) for some integer \( p \geq 1 \). The smallest such \( p \) is called the period of \( z \). If \( p = 1 \), \( z \) is called a fixed point. Also \( z \) is eventually periodic if \( N^{k+p}(z) = N^k(z) \) for positive \( p, k \).

Let \( z \) have period \( p \). Then

\[
\lambda = (N^p)'(z)
\]

is called the eigenvalue of \( z \). Note that

\[
(N^p)'(z_0) = pN^{p-1}(z_0)N'(z_0) = p(p-1)N^{p-2}(z_0)
\]

(He said, by chain rule, this means \( \lambda \) is produce of the derivatives of \( N \) at each point of the orbit of \( z \), and so is an invariant of the orbit. Don’t see this)

Classification of a periodic orbit:

- super attraction: \( |\lambda| = 0 \)
- attraction: \( |\lambda| < 1 \)
- neutral: \( |\lambda| = 1 \)
- repelling: \( |\lambda| > 1 \)

Using Taylor’s series for \( N \), can show that an attracting fixed point has linear convergence, and at least quadratic for super attracting fixed point. In his figure 1, he shows how we approximate the Julia set: those points which are within 0.0001 of each other after 13 iterates.

PRAGMATICS: we usually iterate Newton until the diff. between successive approx. is less than some fixed value.

3 On Julia Sets

Qualitative theory:

4 Standard Newton

The standard Newton (or Newton-Raphson). It is said that the eigenvalue at a root of order \( k \) is \((k-1)/k < 1 \) and "so" it converges linearly here [?].

5 Relaxed Newton

If \( g(x) \) has root of multiplicity \( m \geq 1 \), let \( f(x) = \sqrt[m]{g(x)} \). Then \( f'(x) = \frac{1}{m}g(x)(-m+1/m)g'(m) \) and so \( N_f(x) = x - f(x)/f'(x) = x - \frac{mg(x)}{g'(x)} \).

WHAT IF YOU DO NOT KNOW \( m \)? DOES IT WORK?
§5. Multiple Roots

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6 Method

If \( g(x) \) is a polynomial with simple or multiple roots, then the rational function \( g(x)/g'(x) \) has a simple root for each root of \( g(x) \).

Let us applying Newton’s method to \( f(x) = g(x)/g'(x) \):

\[
f'(x) = (g'(x)^2 - g(x)g''(x))/g'(x)^2 = 1 - (g(x)/g''/g^2).
\]

Then \( f/f' = (g'/g')(g^2/(g^2 - gg'')) = (g'/g' - gg''). \) Therefore,

\[M(x) = x - \frac{g(x)g'(x)}{g'(x)^2 - g(x)g''(x)}.
\]

This method is therefore a sure fire method!

7 Relaxed Newton

If \( g(x) \) has root of multiplicity \( m \geq 1 \), let \( f(x) = \sqrt[m]{g(x)} \). Then

\[f'(x) = \frac{1}{m^2}g(x)(m-1)/m \quad \text{and so}
\]

\[N_f(x) = x - f(x)/f'(x) = x - \frac{mg(x)}{g'(x)}.
\]

This is called the relaxed Newton method.

What happens at a root \( x \) of mult. \( k \)? Compute its eigenvalue at \( x \): it is \( 1 - (m/k) \). So if \( m > 2k \), the eigenvalue has \( |\lambda| > 1 \) and \( x \) will be repelling. If \( m = 2k \), the root \( x \) is neutral. At roots of mult. \( k = m \), then \( x \) will converge quadratically. At \( x = \infty \), eigenvalue is \( d/(d - m) \) where \( d \) is degree of \( g \). This will always be repelling.

8 Method

If \( g(x) \) is a polynomial with simple or multiple roots, then the rational function \( g(x)/g'(x) \) has a simple root for each root of \( g(x) \).

Let us applying Newton’s method to \( f(x) = g(x)/g'(x) \):

\[
f'(x) = (g'(x)^2 - g(x)g''(x))/g'(x)^2 = 1 - (g(x)/g''/g^2).
\]

Then \( f/f' = (g'/g')(g^2/(g^2 - gg'')) = (g'/g' - gg''). \) Therefore, we obtain the Newton iterator for multiple roots:

\[M(x) = x - \frac{g(x)g'(x)}{g'(x)^2 - g(x)g''(x)}.
\]

But \( g(x)/g'(x) \) will have poles those roots of \( g'(x) \) that are not roots of \( g(x) \). Let \( w \) be such a root of \( g'(x) \). Gilbert says these are fixed points of \( M(x) \). \[Pf?\]

Gilbert says that \( g'(w) = 0 \) and \( g(w) \neq 0 \), and \( M'(w) = 2 \). Therefore, it seems, these are always repelling. \[I need to understand this connection!\]

Draw color map of the complex plane for this iterator were \( g(x) = x^2(x^3 - 1) \). Gilbert has such a picture.