Lecture 7

Transcendental Computation

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Overview

We give a brief introduction to transcendental number theory, and issues of transcendental computation. Then we describe a recent result showing a first non-trivial transcendental geometric computation that is computable in the EGC sense.

- 0. Review
- I. Intro to Transcendental Number Theory
- II. A Solved Problem that Isn’t: Shortest Path amidst Discs
0. REVIEW
• REMEMBER: the prize for best exercises

• SHOW: If a reduced rational \( p/q \) is the zero of an integer polynomial \( A(X) = \sum_{i=0}^{m} a_i X^i \) then \( q | a_m \) and \( p | a_0 \)

  * Corollary: if \( p/q \) is algebraic integer, then \( q = 1 \)
  * Corollary: \( \sqrt{2} \) is irrational
What have we learned so far?

- EGC is an effective method to achieve robust numerical algorithms
- The central problem of EGC are the ZERO PROBLEMS
- EGC can be achieved for all algebraic problems
- This lecture: Which non-algebraic problems can we solve?
I. Transcendental Numbers
Introduction

- What is between $\mathbb{A}$ and $\mathbb{R}$?
  * DEFINE: A transcendental number is a non-algebraic number.
  * Is $e$ and $\pi$ algebraic?
  * This is the topic of transcendental number theory

- Easier questions
  * Are there any transcendental numbers? Yes (Cantor)
  * Is $e$ rational?
  * Whiteboard Aside: Proof that $e$ is irrational
  * Whiteboard Aside: Proof that $e$ is not quadratic irrational
Louisville’s Theorem (1844)

- If $\alpha$ is algebraic of degree $m > 1$ then for all $p/q \in \mathbb{Q}$, $|\alpha - (p/q)| > Cq^{-2}$
- Proof: let $A(X)$ be minimal polynomial of $\alpha$
- Then $q^{-m} \leq |A(p/q)| = |A(p/q) - A(\alpha)| = |(p/q) - \alpha| \cdot |A'(\beta)|$
- But $|A'(\beta)| \leq C$ for some constant depending on $\alpha$

Corollary: $\sum_{n=1}^{\infty} 2^{-n!}$ is transcendental
- Proof: take $q = 2^{n!}$ for sufficiently large $n$

Progress is slow:
- Hermite 1873, $e$ is transcendental
- Lindemann 1882, $\pi$ is transcendental
* Roth 1955 (culmination of Thue, Siegel)
* Gelfond Schneider: $e^\pi$ is transcendental. But is $\pi^e$?
PART II. A SOLVED-PROBLEM THAT ISN’T

(Joint with E.Chien, S.Choi, D.Kwon, H.Park)
Shortest Path Amidst Disc Obstacles

- **Given:** Points $p, q \in \mathbb{R}^2$ and a collection $S$ of discs
- **Find:** shortest path from $p$ to $q$ which avoids the obstacles in $S$
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Standard Solution: Reduce to Dijkstra’s Algorithm

• Feasible paths: \( \mu = \mu_1; \mu_2; \cdots; \mu_k \)
  * \( \mu_i \) is a straightline segment iff \( \mu_{i+1} \) is an arc
  * Straightline segments are common tangents to 2 discs

• Apply Dijkstra’s shortest path algorithm to a combinatorial graph \( G = (V, E) \)

• Size of \( G \) is \( O(n^2) \) and algorithm is \( O(n^2 \log n) \).
What is Wrong?

- Real RAM model assumed!
- Length of a feasible path is

\[ d(\mu) = \sum_{i=1}^{k} d(\mu_i) = \alpha + \sum_{i=1}^{m} \theta_i r_i \]  \hspace{1cm} (1)

* \( \alpha \geq 0 \) is algebraic
* \( 0 < r_1 < \cdots < r_m \) are distinct radii of discs
* \( \theta_i \) is total angle (in radians) around discs of radii \( r_i \)
Is it really Transcendental?

- E.g., if $\theta = \pi$, then transcendental.

- **LEMMA**: $\cos \theta_i$ is algebraic

- **COROLLARY (Lindemann)**: A non-zero $\theta_i$ is transcendental
Approach for Comparing Lengths

• Let \( d(\mu) = \alpha + \theta \), and \( d(\mu') = \alpha' + \theta' \)
  * E.g., all discs have unit radius

• **LEMMA:** \( d(\mu) = d(\mu') \) iff \( \alpha = \alpha' \) and \( \theta = \theta' \)

• Hence, we need to ability to add arc lengths
Representation of Arc Lengths

- Let $A$ be a directed arc of a circle $C$
  
  * Represent $A$ by $[C, p, q, n]$.  

\[ Val[C, p, q, r] = (\phi(p, q) + n\pi)r \]
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Representation of arc length by \([C, p, q, n]\).
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Addition of Arc Lengths

- Let $A = [C, p, q, n]$ and $A' = [C', p', q', n']$
  - Say $A$ and $A'$ are compatible if $r(C) = r(C')$ and $q - o(C) = \pm (p' - o(C'))$
  - Special case: line $qp'$ is common tangent
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Addition of Arc Lengths

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  - Say $A$ and $A'$ are compatible if $r(C) = r(C'')$ and $q - o(C) = \pm (p' - o(C''))$
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![Diagram showing the addition of arc lengths with points $C, p, q, n, C', p', q', n'$ and angles $\theta, \theta'$, and $\theta''$.]
Addition of Arc Lengths

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  - Say \( A \) and \( A' \) are compatible if \( r(C) = r(C') \) and \( q - o(C) = \pm (p' - o(C')) \)
  - Special case: line \( qp' \) is common tangent
Decidability

- **THEOREM**: Shortest Path for unit disc obstacles is computable.

- **Extensions**:
  * When Radii of discs are “commensurable”
  * Complexity Bound?
  * Baker’s Linear Form in Logarithms:
    \[
    \left| \alpha_0 + \sum_{i=1}^{n} \alpha_i \log \beta_i \right| > B
    \]

- **THEOREM**: Shortest Paths for algebraic discs is computable.
THEOREM: Shortest Paths for rational discs is in single exponential time.
Conclusions

• First computability result for a (combinatorially non-trivial) transcendental computational problem

• Positive Result from Transcendental Number Theory!
  * Also: Lyapunov (1955)

• Open Problems:
  * Extend to ellipse obstacles
  * Extend to sphere obstacles

• Other examples of transcendental problems
  * Helical motion in robot motion planning
EXERCISES

• Assume \( n \) is not a square. Generalize the usual proof for \( n = 2 \) to show \( \sqrt{n} \) is irrational when \( n \) is even
  * Try to extend to odd \( n \)

• Locate the zero problem for the following:
  * There is a point \( p \) that is rotating with constant angular velocity about the origin \( O \).
  * A unit disc \( D \) is translating with known constant velocity.
  * You want to decide whether \( p \) collides with \( D \)
REFERENCE

• “Shortest Paths for Disc Obstacles is Computable”

“A rapacious monster lurks within every computer, and it dines exclusively on accurate digits.”
THE END