Final Project, part II

Project Organization This is the second part of the final project, containing the remaining three problems. For an honors credit, finish all problems. Otherwise, you don’t need to do Problem 1, 6.

5. Background information. For wave propagation in a cylindrical fiber optic cable, the time harmonic wave function \( u(x, y, z, t) \) has the form

\[
u(x, y, z, t) = \phi(x, y, z)e^{i\omega t}
\]

where \( \omega \) is the temporal frequency of the propagating wave.

The function \( \phi(x, y, z) \), also referred to as the wave function, usually assumes complex values. Each complex number has its real and imaginary parts, or it has its amplitude and phase. At a point \((x, y, z)\) in three dimensions, the complex number \( \phi(x, y, z) \) represents the amplitude and phase of the propagating wave. The function \( \phi(x, y, z) \) satisfies the Helmholtz equation

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 (1 + q(x, y, z)) \psi = 0,
\]

written in the cylindrical coordinates, as opposed to the standard Cartesian coordinates. \( z \) is in the direction of the length of the cable to suit our cable geometry, whereas \((r, \theta)\) is the polar coordinates for the xy-plane or the cross section of the cable. Let \( c_0 \) be the speed of light in vacuum; \( k \) is known as the wave number, a constant defined by

\[
k = \frac{\omega}{c_0}
\]

The function \( q \) is identically zero if the cable is absolutely empty – vacuum. We would not, however, let that happen if we want to fill it with some medium, which is transparent to lights, and which may have desirable properties to lift the efficiency and reduce noise in the transmission of lights, which carry our signals – one of the life lines in the information age. Different filling materials will give rise to different \( q \) values. As a function, \( q(x, y, z) \) in a standard cable is independent of \( z \), so that it becomes \( q(x, y) \). Furthermore, A regular, cylindrical cable is not only circular in its cross section, but also cylindrical in its content, namely, \( q(x, y) \) is dependent only on

\[
r = \sqrt{x^2 + y^2}
\]

where the origin \((x, y) = (0, 0)\) lies on the center of the cable. In particular \( q(x, y) \) is independent of \( \theta \); thus

\[
q(x, y, z) = q(x, y) = q(r)
\]

Similarly, the wave propagating in the cable is also required to be independent of \( \theta \), that is

\[
\psi(x, y, z) = \psi(r, z)
\]

Moreover, if the cable is coated with a layer of metal on its surface – this is used to simplify our problem; the actual wave guide is usually not coated with metal – then the wave will simply repeat itself in the \( z \) direction without decay in its amplitude:

\[
\psi(x, y, z) = \psi(r, z) = \phi(r)e^{i\mu z}
\]

with the repeatability constant \( \mu \), as well as \( \phi(r) \), to be determined. Our job here is to determine them.
The differential equation to solve. Substituting (5), (7) into (2), we observe that

$$\frac{\partial^2 \psi}{\partial \theta^2} = 0, \quad \frac{\partial^2 \psi}{\partial z^2} = -\mu^2 \phi(r) e^{i\mu z}$$

(8)

and (2) is simplified to give

$$\phi''(r) + \frac{1}{r} \phi'(r) + \left[k^2(1 + q(r)) - \mu^2\right] \phi(r) = 0, \quad r \in (a, b)$$

(9)

Let $a = 0$ denote the center of the cable cross section, and let $b > 0$ be given as the diameter of the cable. The metal coating makes the light wave vanish on the surface of the cable, $\phi(b) = 0$. Furthermore, the symmetry of the wave function $\psi$ (it is independent of $\theta$) around the center implies that $\phi'(a) = 0$. We therefore rewrite (9) and put it together with the boundary values

$$\frac{1}{r}[r \phi'(r)]' + [k^2(1 + q(r)) - \mu^2] \phi(r) = 0, \quad r \in (a, b)$$

(10)

$$\phi'(a) = 0,$$  

(11)

$$\phi(b) = 0$$  

(12)

Given $k$ and $q$, this is the ODE with its boundary values – the equations we want to solve for $\mu$ and $\phi(r)$.

An eigenvalue problem. The equations (10), (11), (12) define an eigenvalue problem for $\mu$ and $\phi(r)$. In order to solve it, we need to discretize (10), just as what we've done for Problem 4. The result is an algebraic eigenvalue problem of the form

$$A \phi = \lambda \phi$$

(13)

with $A$ some $n$-by-$n$ matrix, $\phi$ now is considered as an $n$ vector. You need to find out and construct the matrix $A$ and solve (13) by feeding $A$ to the Matlab function `eig`.

Let me now describe how $\lambda$ is related to $\mu$. There are multiple possible values for $\mu$ and the corresponding eigenfunction $\phi(r)$. For consideration of efficiency of cable transmission (I'll not say what exactly it is), we need to find two or three largest possible values for $\mu$, which turn out to be closest to $k$. Let

$$\lambda = \mu^2 - k^2$$

(14)

which appears in (13). Your job is to find two or three smallest eigenvalues of (13).

Statement of work. The introduction may be complicated; what you need to do is quite simple. We first provide the parameters and functions required for (10)–(12)

$$a = 0, \quad b = \pi, \quad k = 20.357, \quad q = -0.6 e^{-2r^2}$$

(15) \hspace{1cm} (16) \hspace{1cm} (17) \hspace{1cm} (18)

Now, here are the 4 steps you need to follow and complete.

(a) Find matrix $A$ by following the steps given in Problem 4, where $f(r) = \lambda \phi(r)$. Print $A$ for the case of $n = 5$.

(b) Check and show the rate (or order) of convergence of your code by supplying a test $f(r) = 1 + \cos(r)$ and with $k = 2$. You need to choose suitable number of points $n_j$, $j = 1, 2, 3$ to do it, and check at all the $n_1$ equispaced points. The order must be at least 2.
(c) Find a suitable \( n \) to solve the eigenvalue problem (13) so that each of the three smallest eigenvalues is accurate to at least three digits. You can do so by increasing \( n \) until the third digit stops changing, meaning that it has converged to three digits. Don’t have misgivings by thinking that it may converge and then diverge and then eventually converge. Yeah, I think I heard that before; it’s all our imagination, or somebody’s imagination. As someone else remarked: “Nature does not play games with us.” Print the three smallest eigenvalues.

(d) Plot absolute value of \( \phi(r) \) in \([a, b]\) which is the amplitude of the wave on the cross section of the cable.

(e) Extra credit. Repeat (a)–(d) for

\[
q = 0.6e^{-2r^2}
\]  

(19)

**Note,** It is not difficult for us to extend the metal coated case to the no coating case where the boundary condition need to be the so-called radiation condition; we will not do it here. But once you have the code for the radiation condition, you can experiment with the \( q \) function, which is the inhomogeneity in the speed of light of the cable material, and see what kind of shape will give you the desired \( \mu \) and also minimize the loss of energy via the radiation through the surface of the cable, now that it is not metally coated. \( q(r) > 0 \) give rise to lower speed than of light than \( c_0 \) – the speed of light in vacuum. \( q(r) > 0 \) is the opposite. A more systematic way of designing \( q \) function to minimize loss of energy in transmission cable is a current, active research topic. It is referred to as optimal design problem.

6 Sometimes we need to integrate a function that is singular, that is, it’s value may not stay bounded. In almost every natural problem that is described by a differential equation, there is an corresponding integral equation which describes the same problem, and almost every integral equation involves the integration of a singular function or functions. Thus, the ability to integrate singular functions accurately is one of the most important basic requirements for the solution of many natural problems; as we have seen from the quadrature formula, (numerical) integration is no more than taking an inner product.

(a) Find the exact value (see page 360) of

\[
\int_0^1 \cos(y) \sqrt{y} \, dy
\]

(20)

(b) Use \( n_1 = 30, n_2 = 40, \) and \( n_3 = 50 \) Gaussian points to integrate (20) numerically. Print the three values and the rate of convergence (give the order you find numerically).

(c) Use the coordinate transformation

\[
y = f(x), \quad dy = f'(x) \, dx
\]

(21)

to rewrite the integral in terms of \( x \). Then apply the composite trapezoidal quadrature rule to compute the integral to 10 digits, with \( f(x) \) chosen as our filter you designed in the midterm project for \( m = n = 5 \). Find the minimum number \( M \) of points for the trapezoidal quadrature to yield accurate 10 digits. Provide the numerical value of the integral for this \( M \).

(d) Estimate the minimum number of points \( M \) to gain accurate 10 digits if you were to use the Gaussian quadrature rule. Hint: Use the expression (3) in Part I, or equivalently use

\[
A(h) = A(0) + a \, h^\alpha
\]

(22)

to estimate \( h = 1/M \) once you know \( a \) and \( \alpha \).