Midterm Project 1.2

Objective: Least-squares solution of under-determined linear system with the “physically relevant norms”. Optimal design of narrow and wide bandwidth digital filters and high order quadrature formulae.

A filter $f$ is required to be a smooth transition from 0 to 1, to be monotone ($f' \geq 0$), and to have vanishing derivatives up to certain order at the two end points $a = -\pi$, $b = \pi$. If done correctly, the function

$$f_3(x) = \frac{1}{2\pi}(x + \pi) + \sum_{j=1}^{n} c_j \sin(jx)$$

will satisfy the requirements (except $f' \geq 0$) with suitably chosen coefficients $c_j$. In the first part of the project, you have set up an m-by-n linear system for $c_j$, solved it with $m = n = 5$ and got a solution which gives rise to an $f_3$ that happens to be monotone.

There is no reason that we must choose $m = n$ and get a square linear system to which the solution is unique. On the contrary, we prefer $m < n$ so that the linear system is under-determined: there are more degrees of freedom than the number of equations. When this happens, the solution is not unique, and therefore there is freedom to choose a solution with desirable properties.

Question 1. Let $D \in \mathbb{R}^{m \times n}$ be invertible. Describe with no more than 40 words (plus necessary formulae) a procedure which uses SVD to solve our problem $A \cdot c = b$ and simultaneously minimizes the 2-norm of $x = D^{-1} \cdot c$. Hint: Consider the equation $A \cdot D^{-1} \cdot D \cdot c = b$. Note: for this problem to make practical sense, it is assumed that $m < n$ (what happens if $m = n$).

Question 2. Find out $D$ if we want to solve our problem $A \cdot c = b$ and minimize the 2-norm of $f_3'$, go to the end of this handout for details on the 2-norm of a continuous function.

Question 3. For $m = 5$, $n = 8$, solve our problem $A \cdot c = b$ and minimize the 2-norm of $f_3^{[2m+2]}$. Show the condition number. Plot $f_3$ so constructed with 32 equispaced points in $[-\pi, \pi]$.

Question 4. For $m = 5$, $n = 31$, $\mu = 0.01$, solve our problem $A \cdot c = b$ and minimize $||f_3^{[4]}||_2^2 + \mu||f_3^{[4]}||_2^2$. Show the condition number. Plot $f_3$ so constructed with 80 equispaced points in $[-\pi, \pi]$.


Question 5. For $m = 10$, $n = 23$, $\mu = 10^{-3}$, modify your code so that it runs the vpa operations with digits(50) to solve our problem $A \cdot c = b$ and minimize $||f_3^{[5]}||_2^2 + \mu||f_3^{[5]}||_2^2$. Show the condition number. Plot $f_3$ so constructed with $k = 80$ points $x_j = -\pi + (j - 1/2)h$, $h = 2\pi/k$ in $(-\pi, \pi)$. Print out in double precision values of $f_3$ and $f_3'$ at $x_1, x_2, \ldots, x_n$.

Remark. An observation useful for debugging; the values of $f_3$ and $f_3'$ should be positive and very small near $x = -\pi$. If they are not, it means that there is a bug and it’s likely that vpa operations are not implemented correctly or fully with 50 digits. This calculation is relatively cpu time intensive so first experiment with smaller $m$ and $n$ and verify your result by comparing it with what you’ve done with double precision arithmetic.
The 2-norm of a function $f$ in $[a, b]$ is defined by the formula
\[
\|f\|_2 = \left( \frac{1}{b-a} \int_{a,b} f^2(x) \, dx \right)^{\frac{1}{2}} ;
\] (2)
therefore it can verify that a sine or cosine series
\[
f(x) = \sum_{j=1}^{n} a_j \sin(jx), \quad \text{or} \quad f(x) = \frac{1}{2} a_0 + \sum_{j=1}^{n} a_j \cos(jx)
\] (3)
in $[a, b] = [-\pi, \pi]$ will have the 2-norm (up to a factor of 2)
\[
\|f\|_2 = \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}, \quad \text{or} \quad \|f\|_2 = \left( \sum_{j=0}^{n} |a_j|^2 \right)^{\frac{1}{2}}
\] (4)