Assignment 7

Exercise 1
Problem 6.4, Page 279.

Exercise 2
(Extra) Suppose that $a(x), u(x)$ are sufficiently smooth. Denote by $\Delta_0$ the standard central difference operator, namely, $\Delta_0a(x) = a(x + h/2) - a(x - h/2)$. Show that
(a) $$(a(x)u'(x))' = \frac{1}{h^2}\Delta_0(a(x)\Delta_0u(x)) + O(h^2);$$
(b) $$(a(x)u'(x))' = \frac{1}{2h^2}\{ [a(x+h)+a(x)] [u(x+h)-u(x)] - [a(x)+a(x-h)] [u(x)-u(x-h)] \} + O(h^2).$$

Exercise 3
Suppose that $a, b, f$ are sufficiently smooth functions, and that $a, b$ are also positive. Consider finite difference solution of the boundary value problem

$$-(a(x)u'(x))' + b(x)u(x) = f(x), \quad x \in (0,1),$$
$$u'(0) = 0,$$  \hspace{1cm} \text{(3)}
$$u'(1) + \gamma u(1) = \beta$$  \hspace{1cm} \text{(4)}

on the equispaced grid $x_j = jh, j = 0,1,\ldots,m, h = 1/m$.
(a) Use (2) to write down second order finite difference approximation of the equation (3) at the interior grid points $x_j, j = 1,2,\ldots,m-1$;
(b) Use the second order forward difference scheme

$$u'(x) = \frac{1}{h}[\Delta_+ - 0.5\Delta_2^2]u(x) + O(h^2); \quad \text{where } \Delta_+ u(x) = u(x+h) - u(x)$$
\hspace{1cm} \text{(6)}

to construct approximation of the left boundary condition (4) (this is a finite difference equation for the first grid point $x_0$);
(c) Similarly use the second order backward difference approximation

$$u'(x) = \frac{1}{h}[\Delta_- + 0.5\Delta_2^2]u(x) + O(h^2); \quad \text{where } \Delta_- u(x) = u(x) - u(x-h)$$
\hspace{1cm} \text{(7)}

to construct approximation of the right boundary condition (5) (this is a finite difference equation at the last grid point $x_m$);
(d) Given

$$a(x) = 1 + x \ln(2+x),$$
$$b(x) = 1/(2+x),$$
$$f(x) = (1+x^2)(1+\cos(15x)),$$
$$\gamma = 1, \beta = 2,$$  \hspace{1cm} \text{(10)}
solve this linear system of \(m+1\) equations for \(m+1\) unknowns \(u_j, j = 0, 1, \ldots, m\) for \(m = 40, 80, 160\). Check and show rate of convergence (note that the linear system is tridiagonal except at the first and the last rows); (e) Plot the numerical solution in \([0, 1]\) for \(m = 160\).

**Exercise 4**

Use the standard second order, five-point stencil, finite difference scheme to solve the Poisson equation

\[
-\nabla^2 u(x, y) = f(x, y), \quad (x, y) \in [0, \pi] \times [0, \pi]
\]  

(12)

subject to the Dirichlet boundary conditions

\[
\begin{align*}
u(0, y) &= g_l(y), \quad y \in [0, \pi], \\
u(\pi, y) &= g_r, \quad y \in [0, \pi], \\
u(x, 0) &= g_b, \quad x \in [0, \pi], \\
u(x, \pi) &= g_t, \quad x \in [0, \pi].
\end{align*}
\]  

(13) \hspace{1cm} (14) \hspace{1cm} (15) \hspace{1cm} (16)

where the subscripts \(l, r, b, t\) represent left, right, bottom, and top.

(a) Read and understand this paragraph only, do nothing else. Let’s first consider the simplest case that is still general enough to see the structure of the discretization: \(n = 5, \ h = \pi/n\). In this case, there are \(6 \times 6 = 36\) mesh points all together: \((x_i, y_j) = (i h, j h), 0 \leq i, j \leq 5\). There are thus \(4 \times 4 = 16\) interior mesh points, and \(4 \times 5 = 20\) mesh points on the boundary. There are \(16\) unknowns

\[
u_{ij} = u(x_i, y_j), \quad 1 \leq i, j \leq 4
\]  

(17)

and we want to re-organize them as a vector of length 16, or \((n - 1)^2\)

\[
u^h = \begin{bmatrix} u_{1,1}^h \\ u_{1,2}^h \\ u_{2,1}^h \\ u_{2,2}^h \\
\end{bmatrix}, \quad \text{where} \quad u_j^h = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \end{bmatrix},
\]  

(18)

In other words, we have \(u^h(k) = u_{ij}\) with \(k = (n - 1) \cdot (j - 1) + i\). Note that the vector \(u^h\) is now indexed by a single integer \(k\) whereas the vector \(u\) is indexed by two integers \((i, j)\); as \(i\) and \(j\) range from 1 to \(n - 1\), \(k\) ranges from 1 to \((n - 1)^2\).

(b) Read and understand this paragraph only, do nothing else. The unknown vector \(u^h\) satisfies the linear system of equations with an error term \(O(h^2)\)

\[
A \ u^h = f^h + g^h + O(h^2),
\]  

(19)

where the vector \(u^h\) still assumes the exact values of the function \(u(x, y)\) at the \((n - 1)^2\) interior mesh points. Abusing the notation slightly, we denote the approximate (numerical) solution also by \(u^h\) which satisfies the linear system of equations

\[
A \ u^h = f^h + g^h
\]  

(20)

where

\[
f^h(k) = f(x_i, y_j), \quad 1 \leq i, j \leq 4, \quad 1 \ k = (n - 1) \cdot (j - 1) + i
\]  

(21)

(c) You are now asked to do something. What are the dimensions of \(A, \ f^h, \ g^h\)

(d) Partition \(A\) as 4-by-4 blocks (in general, \((n-1)\)-by-\((n-1)\) blocks)

\[
A = \frac{1}{h^2} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix},
\]  

(22)
What are the dimensions of each of the blocks $A_{ij}$. Note that this is a block-tridiagonal matrix with

$$A_{ii} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}, \quad i = 1 : 4; \quad A_{ij} = -I_4, \quad |i - j| = 1; \quad A_{ij} = 0 \quad \text{otherwise} \quad (23)$$

(e) Partition the vectors $f^h$, $g^h$ as 4-by-1 blocks. What are the dimensions of the blocks. Determine the entries of each block.

(f) Implement the resulting linear system (20) on a computer for a general $n$ value.

(g) For $n = 6, 12, 24$, solve for $u^h$ with $f(x) = 0$ and subject to the Dirichlet boundary conditions

$$g_b(y) = \sin(\alpha y), \quad g_r = \sin(\beta y), \quad g_0(x) = g_I(x) = 0. \quad (24)$$

Choose $\alpha = 1$ and $\beta = 1$ to check order of convergence (it should be second order in $h$). Use $n = 12, 16, 22$ to check order if necessary.

(h) Use a suitable $n$ value to produce a numerical solution $u^h$ for $\alpha = 15, \beta = 7$. Make a surface plot of the corresponding vector $u_{ij} = u^h(k)$ with $k = (n - 1) \cdot (j - 1) + i$.

(i) Finally, for those with adequate exposure to PDEs, explain the behavior of the (numerical) solution.

**Exercise 5**

**Optional.** Under the conditions of Exercise 3, consider finite difference solution of the initial-boundary value problem for the diffusion equation

$$u(x,t) = (a(x)u'(x,t))^\prime - b(x)u(x,t) + f(x), \quad x \in (0,1), \quad (25)$$

$$u(x,0) = \begin{cases} 1 - \cos(4\pi x), & x \in (0,0.5) \\ 1 - \cos(8\pi x), & x \in (0.5,1) \end{cases} \quad (26)$$

$$u'(0,t) = 0, \quad (27)$$

$$u'(1,t) + \gamma u(1,t) = \beta \quad (28)$$

Let $u_h = [u(x_0), u(x_1), \ldots, u(x_n)]^\prime$ be the numerical solution of Exercise 3. Denote by $U_h(t) = [u(x_0,t), u(x_1,t), \ldots, u(x_n,t)]^\prime$ be the numerical solution of Exercise 5. Let $Au_h = f_h$ be the discrete system obtained from Exercise 3 with the spatial grid $x_j = jh, \ j = 0,1,\ldots, m, \ h = 1/m$. Then $U_h(t)$ satisfies the semi-discretized diffusion equation – a system of ODEs for the vector $U_h(t)$

$$\frac{d}{dt}U_h(t) = -A \cdot U_h(t) + f_h \quad (29)$$

to the second order of $h$.

(a) Solve the ODEs (29) with the forward Euler’s method with $\Delta t = 2h^2$ and observe that the marching in t-steps is not stable.

(b) Same as above but with $\Delta t = 0.5h^2$ and see if it is stable. If not, try $\Delta t = 0.2h^2$.

(c) Solve the ODEs (29) with the backward Euler’s method with $\Delta t = 2h^2$ and observe that the marching in t-steps is stable.

(d) Same as above but with $\Delta t = h$ and observe that the marching in t-steps is stable.

(e) Solve the ODEs (29) with the trapezoidal method with $\Delta t = h$. Check order of convergence at $t = 0.5$ with $m = 40, 80, 160$. It should be convergent with second order.

Note: Solve each case till $t = 2$. See (8)-(11) for the parameters. For the forward Euler and trapezoidal method, use sparse LU factorization for the matrix $A$ once for a fixed $m$ value, and then do the back solve for each t-step.

(f) Plot $U_h(t)$ for $m = 80$ and $t = 2$ and compare it with the plot of Exercise 3.

(g) Comment on the efficiencies of the forward, backward Euler, and the trapezoidal methods.