Tatonnement in Ongoing Markets of Complementary Goods

Yun Kuen CHEUNG (Marco) ¹
Richard COLE ¹
Ashish RASTOGI ²

¹ - Courant Institute of Mathematical Sciences, NYU
² - Goldman Sachs & Co., New York
Take-Home Message

We prove that tatonnement, with non-equilibrium trade, leads to fast convergence toward market equilibrium in a class of ongoing markets with either

— complementary goods, or
— mixture of substitutes and complements

when demand elasticity and income elasticity are suitably bounded.
Overview of Talk

1. Problem Definition

2. Tatonnement Price Update Rule

3. Results

4. Analysis
Market Equilibrium

- In general, hard to compute.
Market Equilibrium

- In general, hard to compute.
- And, is it reached?
Market Equilibrium

- In general, hard to compute.
- And, is it reached?
- Papadimitriou (2002): If so, (near-)equilibrium prices ought to be P-time computable.
Market Equilibrium

- In general, hard to compute.
- And, is it reached?
- Papadimitriou (2002): If so, (near-)equilibrium prices are ought to be P-time computable.
- Cole and Fleischer (2008): And they are surely also being computed within the markets.
Market Equilibrium

- In general, hard to compute.
- And, is it reached?
- Papadimitriou (2002): If so, (near-)equilibrium prices are ought to be P-time computable.
- Cole and Fleischer (2008): And they are surely also being computed within the markets.
- Neither is true in general.
Market Equilibrium

Natural Question:
For what types of markets can we show fast convergence within the market?
Market Equilibrium

- **Natural Question:**
  For what types of markets can we show fast convergence within the market?

- **Within the market:**
  What are appropriate properties for the price update rule?
Appropriate Properties for Price Update Rules [Cole,Fleischer]

- Limited Information (locality)
- Agents follow simple rules
- Asynchrony
- Fast Convergence
Walras (1874): Price adjustment as follows:

**EXCESS SUPPLY**: price decreases
**EXCESS DEMAND**: price increases
Widespread Viewpoint:
Tatonnement is “only” an algorithmic process.
Tatonnement Price Update

- Widespread Viewpoint: Tatonnement is “only” an algorithmic process.

- Our Viewpoint: In fact, for a substantial class of markets tatonnement is a reasonable in-market process with fast convergence toward equilibrium.
Fisher Markets

- each buyer
  - has only money
  - chooses affordable purchase maximizing its utility

- each seller
  - wants only money
  - sells one good
Fisher Markets

**TARGET:** Find equilibrium prices - the prices such that demand = supply for all goods; and find them quickly.
Ongoing Markets [Cole,Fleischer]

- Add notion of time: market repeats daily.
- Trading enabled at disequilibrium.
- Introduce warehouses to handle excess demand/supply.
Overview of Talk

1. Problem Definition

2. *Tatonnement Price Update Rule*

3. Results

4. Analysis
Walras (1874): Price adjustment as follows:

**EXCESS SUPPLY:** price decreases

**EXCESS DEMAND:** price increases

- \( x_i \): instantaneous demand for good \( i \)
- \( w_i \): supply of good \( i \)
- \( v_i \): current amount of good \( i \) in warehouse \( i \)
- \( v_i^* \): the target amount of good \( i \) in warehouse \( i \)
Tatonnement Price Update

- \( x_i \): instantaneous demand for good \( i \)
- \( w_i \): supply of good \( i \)
- \( v_i \): current amount of good \( i \) in the warehouse
- \( v_i^* \): the target amount of good \( i \) in the warehouse

Version 1:

\[
p'_i \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i}{w_i} \right] \cdot \Delta t \right\}
\]
Tatonnement Price Update

- $x_i$: instantaneous demand for good $i$
- $w_i$: supply of good $i$
- $v_i$: current amount of good $i$ in the warehouse
- $v_i^*$: the target amount of good $i$ in the warehouse

**Version 1:**

$$p'_i \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i}{v_i} \right] \cdot \Delta t \right\}$$

excess demand
Tatonnement Price Update

- $x_i$: instantaneous demand for good $i$
- $w_i$: supply of good $i$
- $v_i$: current amount of good $i$ in the warehouse
- $v_i^*$: the target amount of good $i$ in the warehouse

**Version 1:**

$$p_i' \leftarrow p_i \left\{1 + \lambda \cdot \min \left[1, \frac{x_i-w_i}{w_i}\right] \cdot \Delta t \right\}$$

the time since the last price update
Tatonnement Price Update

- \( x_i \): instantaneous demand for good \( i \)
- \( w_i \): supply of good \( i \)
- \( v_i \): current amount of good \( i \) in the warehouse
- \( v_i^* \): the target amount of good \( i \) in the warehouse

**Version 2:**

\[
p'_i \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i}{w_i} \right] \cdot \Delta t \right\}
\]
Tatonnement Price Update

- $x_i$: instantaneous demand for good $i$
- $w_i$: supply of good $i$
- $v_i$: current amount of good $i$ in the warehouse
- $v_i^*$: the target amount of good $i$ in the warehouse

**Version 2:**

$$p_i' \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i}{w_i} \right] \cdot \Delta t \right\}$$

NOT instantaneous demand, but rather average demand since last update. ($\bar{x}_i \Delta t$ is total demand since last update.)
Tatonnement Price Update

- $x_i$: instantaneous demand for good $i$
- $w_i$: supply of good $i$
- $v_i$: current amount of good $i$ in the warehouse
- $v_i^*$: the target amount of good $i$ in the warehouse

Finally We Use:

$$p'_i \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i - \kappa(v_i - v_i^*)}{w_i} \right] \right\} \cdot \Delta t$$
Tatonnement Price Update

- $x_i$: instantaneous demand for good $i$
- $w_i$: supply of good $i$
- $v_i$: current amount of good $i$ in the warehouse
- $v_i^*$: the target amount of good $i$ in the warehouse

Finally We Use:

$$p'_i \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i - \kappa (v_i - v_i^*)}{w_i} \right] \right\} \cdot \Delta t$$

adjustment to account for warehouse imbalance
Tatonnement Price Update

- $x_i$: instantaneous demand for good $i$
- $w_i$: supply of good $i$
- $v_i$: current amount of good $i$ in the warehouse
- $v_i^*$: the target amount of good $i$ in the warehouse

Finally We Use:

$$p_i' \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{\bar{x}_i - w_i}{w_i} - \kappa (v_i - v_i^*) \right] \cdot \Delta t \right\}$$

small enough parameters $\lambda, \kappa$
Overview of Talk

1. Problem Definition
2. Tatonnement Price Update Rule
3. Results
4. Analysis
Prior Work

- Largely limited to markets with goods that are substitutes.

- Exceptions:
  - Eisenberg-Gale convex program in Fisher market (buyers having any CES utilities)
  - Codenotti et al. (2005): convex program for computing equilibrium in P-time with complements in exchange market (agents having CES utilities with $\rho \geq -1$)
Prior Work

- [Cole and Fleischer] [Cole, Fleischer and Rastogi] achieve fast convergence allowing:
  1. Asynchronous Price Updates
  2. Bounded Size Warehouses
  with market constraints
    a. Weak Gross Substitutes (WGS)
    b. Bounded Demand Elasticity
    c. Bounded Income Elasticity
New Results

Example - CES Utility Functions

\[ u(x_1, x_2, x_3) = \left(2x_1^{\rho} + 3x_2^{\rho} + 5x_3^{\rho}\right)^{1/\rho} \]
Example - CES Utility Functions

\[ u(x_1, x_2, x_3) = (2x_1^\rho + 3x_2^\rho + 5x_3^\rho)^{1/\rho} \]

We show tatonnement converges quickly when buyers have complementary CES utility functions with \(-1 < \rho \leq 0\) (distinct \(\rho\) for each buyer).
New Results

A. Achieve fast convergence allowing:
   1. Asynchronous Price Updates
   2. Bounded Size Warehouses

   with market constraints:
   a. Bounded Demand Elasticity with parameter $\alpha$
      if one price $p_i$ raised by a factor $r$, $x_i' \leq x_i/r^{\alpha}$
   b. Bounded Income Elasticity with parameter $\gamma$
      if all prices raised by a factor $q$, $x_i' \geq x_i/q^{\gamma}$
   c. Markets with complementary goods and
      \[
      \beta := 2\alpha - \gamma > 0
      \]
New Results

A. Achieve fast convergence allowing:

1. Asynchronous Price Updates
2. Bounded Size Warehouses

with market constraints:

a. Bounded Demand Elasticity with parameter $\alpha$

b. Bounded Income Elasticity with parameter $\gamma$

c. Markets with complementary goods and

\[ \beta := 2\alpha - \gamma > 0 \]

For CES with $\rho > -1$, $\alpha = \frac{1}{1-\rho}$, $\gamma = 1$, so $\beta > 0$. 
Overview of Talk

1. Problem Definition

2. Tatonnement Price Update Rule

3. Results

4i. Phase 1 Analysis

3ii. More Results

4ii. Phase 2 Analysis
Intuition for Analysis

- Phase 1: shows fast convergence when warehouse imbalance has modest effect on price update
  \[
  p'_i \leftarrow p_i \left\{ 1 + \lambda \cdot \min \left[ 1, \frac{x_i - w_i - \kappa(v_i - v_i^*)}{w_i} \right] \cdot \Delta t \right\}
  \]

- Phase 2: Used when warehouse imbalance could have substantial impact
  - can guarantee prices are always fairly close to equilibrium
  - assumes warehouses never overflow nor out of stock (shown via a separate argument)
  - amortized analysis
Analysis of Phase 1 — Warehouse
Effect is Relatively Small

\[ G_i \]

\[ \frac{p^*}{f} \quad p^* \quad f p^* \]

\[ f \text{-bounded prices} \]
Analysis of Phase 1 — Warehouse

Effect is Relatively Small

\[
p^*/f, \quad p^*, \quad fp^*
\]

Worst Case: Since goods are complementary, the higher the other prices, the lower the demand for \( G_i \).
Analysis of Phase 1 — Warehouse
Effect is Relatively Small

Worst Case: Since goods are complementary, the higher the other prices, the lower the demand for $G_i$.

This price will shift inward when $\beta := 2\alpha - \gamma > 0$
Overview of Talk

1. Problem Definition
2. Tatonnement Price Update Rule
3. Results
  4i. Phase 1 Analysis
  3ii. More Results
  4ii. Phase 2 Analysis
Adverse Market Elasticity (AME) generalizes demand and income elasticities:

\[
\bar{P} := \left\{ ((1 + \delta)p_1, q_2, \cdots, q_n) \mid \text{for } i \geq 2, q_i \in \left[ \frac{1}{1 + \delta}, (1 + \delta)p_i \right] \right\}
\]

The AME for \( G_1 \) is defined to be

\[
- \max_{\bar{p} \in \bar{P}} \lim_{\delta \to 0} \frac{x_1(\bar{p}) - x_1(p)}{\delta x_1(p)}
\]

Define \( \beta \) to be a lower bound on the AME over all goods and prices.
New Results

B. Achieve fast convergence allowing:

1. Asynchronous Price Updates
2. Bounded Size Warehouses

with market constraints:

a. AME $\beta > 0$

b. A technical assumption

• The $\beta$ in AME matches the $\beta = 2\alpha - \gamma$ in complementary case.
New Results

Example - Nested CES Utility Functions

\[ u(x_1, x_2, x_3, x_4) = \left[ \left( \left( 2x_1^{1/2} + 3x_2^{1/2} \right)^2 \right)^{-1/3} + \left( \left( 5x_3^{1/4} + 7x_4^{1/4} \right)^4 \right)^{-1/3} \right]^{-3} \]
New Results

Example - Nested CES Utility Functions

\[ u(x_1, x_2, x_3, x_4) = \left[ \left( \frac{2x_1^{1/2} + 3x_2^{1/2}}{2} \right)^{2} \right]^{-1/3} + \left[ \left( \frac{5x_3^{1/4} + 7x_4^{1/4}}{4} \right)^{4} \right]^{-1/3} \]^{-3}

First group of substitutes.

Second group of substitutes.
New Results

Example - Nested CES Utility Functions

\[ u(x_1, x_2, x_3, x_4) = \left[ \left( 2x_1^{1/2} + 3x_2^{1/2} \right)^2 \right]^{-1/3} + \left[ \left( 5x_3^{1/4} + 7x_4^{1/4} \right)^4 \right]^{-1/3} \]^{-3} \]

First group of substitutes.

Second group of substitutes.

Any two goods from different groups form a pair of complements.
New Results

Example - Nested CES Utility Functions

\[ u(x_1, x_2, x_3, x_4) = \left[ \left( \left( 2x_1^{1/2} + 3x_2^{1/2} \right)^2 \right)^{-1/3} + \left( \left( 5x_3^{1/4} + 7x_4^{1/4} \right)^4 \right)^{-1/3} \right]^{-3} \]

For 2-nested CES, when outer \( \rho \) satisfies 
\(-1 < \rho \leq 0\) and inner \( \rho \)'s satisfy \( \rho \geq 0 \), AME \( \beta > 0 \). Hence, tatonnement converges quickly when buyers have such 2-nested CES.
New Results

Example - Nested CES Utility Functions

\[ u(x_1, x_2, x_3, x_4) = \left[ \left( \left( 2x_1^{1/2} + 3x_2^{1/2} \right)^2 \right)^{-1/3} + \left( \left( 5x_3^{1/4} + 7x_4^{1/4} \right)^4 \right)^{-1/3} \right]^{-3} \]

For 2-nested CES, when outer \( \rho \) satisfies \(-1 < \rho \leq 0\) and inner \( \rho \)'s satisfy \( \rho \geq 0 \), AME \( \beta > 0 \). Hence, tatonnement converges quickly when buyers have such 2-nested CES.

Tatonnement also converges quickly when buyers have \( N \)-nested CES with suitable constraints on the \( \rho \)'s in each level.
Overview of Talk

1. Problem Definition
2. Tatonnement Price Update Rule
3. Results
   4i. Phase 1 Analysis
   3ii. More Results
   4ii. Phase 2 Analysis
Analysis of Phase 2 — Warehouse

Effect may be Relatively Large

- Amortized Analysis

- We design a potential function which decreases continuously with time and
does not increase at discrete events (price changes).
Analysis of Phase 2 — Warehouse
Effect may be Relatively Large

The Potential Function

\[ \phi := \sum_i \phi_i \]

\[ \phi_i := p_i \left[ \text{span}\{\bar{x}_i, x_i, \tilde{w}_i\} - c_1 \lambda (t - \tau_i) |\bar{x}_i - \tilde{w}_i| + c_2 |\tilde{w}_i - w_i| \right] \]

\( \tau_i \) is the time of the last update to \( p_i \)

\( \text{span}\{a, b, c\} := \max\{a, b, c\} - \min\{a, b, c\} \)

\[ \tilde{w}_i = w_i + \kappa (v_i - v_i^*) \]
Analysis of Phase 2 — Warehouse Effect may be Relatively Large

\[ \phi_i := p_i[\text{span}\{\bar{x}_i, x_i, \tilde{w}_i\} - c_1 \lambda(t - \tau_i)|\bar{x}_i - \tilde{w}_i| + c_2|\tilde{w}_i - w_i|] \]
$\phi_i := p_i[\text{span}\{\bar{x}_i, x_i, \tilde{w}_i\} - c_1 \lambda (t - \tau_i) |\bar{x}_i - \tilde{w}_i| + c_2 |\tilde{w}_i - w_i|]$

decreases continuously with time, whenever there is no price update

Analysis of Phase 2 — Warehouse Effect may be Relatively Large
$\phi_i := p_i \left[ \text{span} \{ \bar{x}_i, x_i, \tilde{w}_i \} - c_1 \lambda (t - \tau_i) |\bar{x}_i - \tilde{w}_i| + c_2 |\tilde{w}_i - w_i| \right]$

When there is a price update to $p_i$, the second term increases to 0, which is offset by the decrease of the first term.
Analysis of Phase 2 — Warehouse Effect may be Relatively Large

\[ \phi_i := p_i[\text{span}\{\bar{x}_i, x_i, \tilde{w}_i\} - c_1 \lambda(t - \tau_i)|\bar{x}_i - \tilde{w}_i| + c_2|\tilde{w}_i - w_i|] \]

The third term accounts for the warehouse imbalance.
Result — Theorem 1

Suppose that AME $\beta > 0$, the prices are $f$-bounded throughout the first day, and in addition that $\kappa, \lambda$ are suitably bounded.

Let $M = \sum_j b_j$ be the daily supply of money to all the buyers.

Then the prices become $(1 + \eta)$-bounded after

$$O \left( \frac{1}{\lambda} \ln f + \frac{1}{\lambda \beta} \ln \frac{1}{\beta} + \frac{1}{\kappa} \log \frac{M}{\eta \min_i w_i p_i^*} \right)$$

(days).

Phase 1

Phase 2
Take-Home Message

We prove that tatonnement, with **non-equilibrium trade**, leads to fast convergence toward market equilibrium in a class of **ongoing markets** with either

— complementary goods, or
— mixture of substitutes and complements

when **Adverse Market Elasticity** is suitably bounded.
Q & A