# Midterm Exam with SOLUTIONS

ANSWER ALL QUESTIONS. THIS IS AN OPEN BOOK, IN-CLASS EXAM. PLEASE WRITE ONLY IN COMPLETE ENGLISH SENTENCES, EVEN IF THEY INVOLVE CENTERED EQUATIONS. UP TO 15% OF POINTS ARE ALLOTTED FOR THE CLARITY OF YOUR ANSWER.

**PROBLEM 1** (Short Questions) [10 points for each part]

Classify each of the following assertions as True or False or Unknown (T/F/U). *Brief justification is needed to get any credit at all.* If "Unknown", say why the relevant theorems don't yield a conclusion. You only need to know results from Chapter 2 in this problem, although you may also quote general results that you know.

(a) co-$NSPACE(3^n) \subseteq DSPACE(6^n)$.

(b) $NSPACE(\sqrt{\log n}) \subseteq DSPACE(\log n)$.

(c) Let $\mu : 2^X \to 2^X$ be monotone. We define $A \subseteq X$ to be **big** if $\mu(A) \subseteq A$, and **small** if $A \subseteq \mu(A)$. Let $A^* \subseteq A$ be the union of all small sets, and $B^* \subseteq X$ be the intersection (not union) of all big sets. We know that $A^*$ and $B^*$ are fixed points of $\mu$. If $C$ is any fixed point of $\mu$, then $B^* \subseteq C \subseteq A^*$.

(d) Let $M$ be a dfa that accepts $A$. Then there is an nfa $N$ that accepts the "reverse language" $A^R = \{w^R : w \in A\}$. Note that $w^R$ is the reverse of $w$, given by $w^R[i] = w[n+1-i]$ where $n = |w|$.

SOLUTION:

(a) Unknown: We know that co-$NSPACE(3^n) = NSPACE(3^n) \subseteq DSPACE(9^n)$, from the Immerman result, and Savitch's result. Based on chapter 2 results, we do not know if $DSPACE(9^n) \subseteq DSPACE(6^n)$. Actually, we know that the latter is false. STILL, we cannot conclude that the result is False, because we do not know if Savitch's result is tight.

REMARK: the Immerman result applies to running complexity, but this for the "nice function" $3^n$ used in this space complexity, the result applies to accepting complexity.

(b) Unknown: The only relation between deterministic and nondeterministic space comes from Savitch's theorem. But this theorem works for space bounds above $\log n$, not $\sqrt{\log n}$.

(c) True: $\mu(C) = C$ implies $C$ is both big and small. This means $C \subseteq A^*$ and $B^* \subseteq C$.

(d) True: Let $M = (Q, \Sigma, \delta, q_0, F)$. Let $N = (2^Q, \delta', F, F')$. Also, $q' \in F'$ iff $q_0 \in q'$. Basically, $N$ keeps track of all the states of $M$ that can reach its "current set of sets".

In more detail: The set $\delta'(q', a)$ is defined to be $\{q \in Q : \delta(q, a) \in q'\}$.

**PROBLEM 2** (CFL) [20 points]

Show that CFL is not closed under complementation.

Besides quoting general results from lecture notes, anything else must to be proved from first principles.

SOLUTION: Let $L = \{a^n b^n c^n : n \geq 0\}$. This is not CFL (use pumping lemma). Consider co-$L$. Let $L_0 = \{a^i b^j c^k : i, j, k \geq 0\}$ and $L_1 = \{a^i b^j c^k : i \neq j, \text{or} \quad j \neq k\}$. Then co-$L$ = co-$L_0 \cup L_1$. You can easily show that co-$L_0$ and $L_1$ are CFL, and hence co-$L$ is CFL.

**PROBLEM 3** (Computability) [20 points]

For languages $A, B \subseteq \Sigma^*$, we say $A$ is **many-one reducible** to $B$, denoted $A \leq_m B$, if there is some total recursive transformation $t$ such that for all $x \in \Sigma^*$, $x \in A$ iff $t(x) \in B$. Show that every r.e. language is many-one reducible to

$$HALT = \{i : i \in W_i\} = \{i : \phi_i(i) \downarrow\}.$$

SOLUTION: Let $A$ be r.e., computed by some TM $M$. For any input $x$, we construct a TM $M_x$ that on input $y$, will halt iff $x \in A$. Let $i(x)$ be index of $M_x$. Therefore, $i(x) \in HALT$ iff $x \in A$.

Our construction shows that $i(\cdot)$ is s total recursive function. Hence $A \leq_m HALT$. Q.E.D.

REMARK: Since $\Sigma$ is arbitrary, we ought to make the adjustment that whenever an $i \in \mathbb{N}$ is used in the above proof, it refers to the word in $\Sigma^*$ which is the $|\Sigma|$-adic notation for $i$.

**PROBLEM 4** (Complexity) [30 points]

Let $A \subseteq \Sigma^*$ and $0, 1 \notin \Sigma$. Show that the following are equivalent:
(a) $A \in NP$
(b) There exists $B \in P$ and a constant $k \geq 1$, such that for all $x \in \Sigma^*$,

$$x \in A \Leftrightarrow (\exists y \in \{0,1\}^*) \left[ |y| \leq |x|^k \wedge xy \in B \right].$$

SOLUTION: ($\Rightarrow$) Let $A$ be accepted by a nondeterministic TM $N$ that accepts in time $n^\ell$ (for some constant $\ell \geq 1$). In this case, every computation path of $N$ of length $n^\ell$ on input of length $|x| = n$ can be encoded by a binary string of length $n^{2\ell}$. Let $k = 2\ell$ and

$$B = \{xy : x \in \Sigma^*, \text{and } y \text{ encodes an accepting computation path of } N \text{ for } y \text{ of length } n^k\}.$$

It is easy to see that $B \in P$ and $x \in A$ iff there is a $y \in \{0,1\}^*$ such that $|y| \leq |x|^k$ and $xy \in B$.

($\Leftarrow$) Suppose $B$ can be accepted by a deterministic TM $M$ that accepts in time $n^\ell$ for some $\ell$. We construct a nondeterministic $N$ to accept $A$: on input $x$, $N$ will guess a binary string $y$ of length $|x|^k$, and then run $M$ on $xy$. The running time of $M$ on $xy$ is $|xy|^\ell = (n + n^k)^\ell$, which is polynomial in $n = |x|$. This proves that $A \in NP$.